STOCHASTIC NONPARABOLIC DISSIPATIVE SYSTEMS MODELING THE FLOW OF LIQUID CRYSTALS : STRONG SOLUTION (Mathematical Analysis of Incompressible Flow)

BRZEZNIAK, ZDZISLAW; HAUSENBLAS, ERIKA; RAZAFIMANDIMBY, PAUL

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ZDZISLAW BRZEŹNIAK, ERIKA HAUSENBLAS, AND PAUL RAZAFIMANDIMBY

1. Introduction

Nematic liquid crystal is a state of matter between that has properties between amorphous liquid and crystalline solid. Molecules of nematic liquid crystals are long and thin, and they tend to align along a common axis. This preferred axis indicates the orientations of the crystalline molecules, hence it is useful to characterize its orientation with a vector field $\mathbf{d}$ which is called the **director**. Since its magnitude has no significance, we shall take $\mathbf{d}$ as a unit vector. We refer to [8] and [12] for a comprehensive treatment of the physics of liquid crystals. To model the dynamics of nematic liquid crystals most scientists use the continuum theory developed by Ericksen [15] and Leslie [23]. From this theory F. Lin and C. Liu [24] derived the most basic and simplest form of dynamical system describing the motion of nematic liquid crystals filling a bounded region $\mathcal{O}\subset\mathbb{R}^n(n=2,3)$. This system is given by

$$\begin{aligned}\mathbf{v}_t+(\mathbf{v}\cdot\nabla)\mathbf{v}-\Delta\mathbf{v}+\nabla p &= -\lambda\nabla\cdot(\nabla\mathbf{d}\otimes\nabla\mathbf{d}), &(1.1)\\ \operatorname{div}\mathbf{v} &= 0, &(1.2)\\ \mathbf{d}_t+(\mathbf{v}\cdot\nabla)\mathbf{d} &= \gamma(\Delta\mathbf{d}+|\nabla\mathbf{d}|^2\mathbf{d}), &(1.3)\\ |\mathbf{d}|^2 &= 1. &(1.4)\end{aligned}$$

Here p represents the pressure of the fluid and $\mathbf{v}$ its velocity. By the symbol $\nabla\mathbf{d}\otimes\nabla\mathbf{d}$ we mean a square $n\times n$-matrix with entries defined by

$$[\nabla\mathbf{d}\otimes\nabla\mathbf{d}]_{i,j}=\sum_{k=1}^n\frac{\partial\mathbf{d}^k}{\partial x_i}\frac{\partial\mathbf{d}^k}{\partial x_j},\quad\text{for any } i,j=1,\dots,n.$$

In the present work we assume that the boundary of $\mathcal{O}$ is smooth and the system stated above is subjected to the following boundary conditions

$$\mathbf{v}=0\text{ and }\frac{\partial\mathbf{d}}{\partial\mathbf{n}}=0\text{ on }\partial\mathcal{O}. \tag{1.5}$$

The vector $\mathbf{n}(x)$ is the outward unit and normal vector at each point x of $\mathcal{O}$.

Although the system (1.1)-(1.5) is the most basic and simplest form of equations from the Ericksen-Leslie continuum theory, it retains the most physical significance of the nematic liquid crystals. Moreover it offers many interesting mathematical problems. In fact, the system (1.1)-(1.5) is basically a coupling of the Navier-Stokes equations (NSEs) and the heat flow of harmonic maps (HFHM) onto 2-dimensional sphere S^2. On the one hand it is a coupling of constrained initial-boundary value problems involving gradient nonlinearities. On other hand, a number of challenging questions about the solutions to Navier-Stokes equations and heat flow of harmonic maps are still opened. Therefore we must encounter difficult problems and we should not expect better results than those obtained for the NSE or HFHM when they are coupled together.

In 1995, F. Lin and C. Liu [24] proposed an approximation of the system (1.1)-(1.5) to relax the constraint $|\mathbf{d}|^2=1$ and the gradient nonlinearity $|\nabla\mathbf{d}|^2\mathbf{d}$. More precisely, they studied the

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following system of equations

$$\begin{aligned}\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v} - \mu\Delta\mathbf{v} + \nabla p &= -\nabla\cdot(\nabla\mathbf{d}\otimes\nabla\mathbf{d}), && (1.6)\\ \operatorname{div}\mathbf{v} &= 0, && (1.7)\\ \mathbf{d}_t + (\mathbf{v}\cdot\nabla)\mathbf{d} &= \Delta\mathbf{d} - \frac{1}{\varepsilon^2}(|\mathbf{d}|^2 - 1)\mathbf{d}. && (1.8)\end{aligned}$$

Problem (1.6)-(1.8) with (1.5) is much simpler than (1.1)-(1.4) with (1.5), but it is still a difficult and interesting problem. Since the pioneering work [24] the systems (1.6)-(1.8) and (1.1)-(1.4) have been the subject of intensive mathematical studies. We refer, among others, to [16, 18, 24, 25, 26, 27, 34] and references therein for the relevant results.

In this paper we are interested in the mathematical analysis of a stochastic version of (1.6)-(1.8). Basically, we will investigate a system of stochastic evolution equations which is obtained by introducing appropriate noise term in (1.1)-(1.4). More precisely we consider a trace class Wiener process W_1 and a standard real-valued Brownian motion W_2. We assume that W_1 and W_2 are mutually independent. We consider the problem

$$d\mathbf{v}(t) + \Big[(\mathbf{v}(t)\cdot\nabla)\mathbf{v}(t) - \Delta\mathbf{v}(t) + \nabla p\Big]dt = -\nabla\cdot(\nabla\mathbf{d}(t)\circ\nabla\mathbf{d}(t))dt + S(\mathbf{v}(t))dW_1, \quad (1.9)$$

$$\operatorname{div}\mathbf{v}(t) = 0, \quad (1.10)$$

$$d\mathbf{d}(t) + (\mathbf{v}(t)\cdot\nabla)\mathbf{d}(t)dt = \Big[\Delta\mathbf{d}(t) - \tfrac{1}{\varepsilon^2}(|\mathbf{d}|^2 - 1)\mathbf{d}\Big] + (\mathbf{d}(t)\times h)\circ dW_2, \quad (1.11)$$

$$|\mathbf{d}(t)|^2 \le 1 \text{ a.e. } (x,t)\in Q\times[0,T], \quad (1.12)$$

where $(\mathbf{d}(t)\times h)\circ dW_2$ should be understood in the Stratonovich sense. In 2-D the vector (or cross) product $h\times\mathbf{d}$ is a scalar that should be understood as follows

$$(h^1e_1 + h^2e_2 + 0e_3)\times(\mathbf{d}^1e_1 + \mathbf{d}^2e_2 + 0e_3) = 0e_1 + 0e_2 + (h^1\mathbf{d}^2 - h^2\mathbf{d}^1)e_3,$$

where (e_1, e_2, e_3) is the canonical basis of $\mathbb{R}^3$.

Our work is motivated by the importance of external perturbation on the dynamics of the director field $\mathbf{d}$. Indeed, an essential property of nematic liquid crystals is that its director field $\mathbf{d}$ can be easily distorted. However, it can also be aligned to form a specific pattern by the help of magnetic or electric fields. This pattern formation occurs when a threshold value of the magnetic or electric field is attained; this is the so called Fréedericksz transition. Random external fields change a little bit the threshold value for the Fréedericksz transition. It has been also shown that with the fluctuation of the magnetic field the decay time of an unstable state diminishes. For these results we refer, among others, to [2, 20, 33] and references therein. In all of these works the effect of the hydrodynamic flow has been neglected. However, it is pointed out in [12, Chapter 5] that the fluid flow disturbs the alignment and conversely a change in the alignment will induce a flow in the nematic liquid crystal. Hence for a full understanding of the effect of fluctuating magnetic field on the behavior of the liquid crystals one needs to take into account the dynamics of $\mathbf{d}$ and $\mathbf{v}$. To initiate this kind of investigation we propose a mathematical study of (1.9)-(1.11) which basically describes an approximation of the system governing the nematic liquid crystals under the influence of fluctuating external forces. To the best of our knowledge our work is the first mathematical work which studies the effect of fluctuating external forces to the system (1.9)-(1.11). We mainly establish the existence of strong solution. Here strong solution is understood in stochastic analysis and in PDEs sense as well. Our results are the stochastic counterparts of the ones obtained by Lin and Liu in [24].

The organization of the present article is as follows. In the first subsection of Section 2 we introduce some notation used throughout this paper. In the very subsection we also state the existence of a unique maximal strong solution to our problem. This maximal solution is global for the two dimensional case. A maximum principle type theorem is proved in the last section of the paper. In Section 3 we establish the existence of local and maximal solution to an abstract nonlinear stochastic evolution equations. The existence of maximal solution stated in the first subsection of Section 2 is a consequence of this general result. In the appendix we recall or prove several results which are used to infer that (1.9)-(1.11) with (1.5) falls within the framework of Section 3.

2. Strong solution of stochastic liquid crystals (SLC)

2.1. Functional spaces and Preparatory lemma. Let $n \in \{2,3\}$ and assume that $\mathcal{O} \subset \mathbb{R}^n$ is a bounded domain with boundary $\partial\mathcal{O}$ of class $\mathcal{C}^\infty$. For any $p \in [1,\infty)$ and $k \in \mathbb{N}$, $\mathbb{L}^p(\mathcal{O})$ and $\mathbb{W}^{k,p}(\mathcal{O})$ are the well-known Lebesgue and Sobolev spaces, respectively, of $\mathbb{R}^n$-valued functions. The corresponding spaces of scalar functions we will denote by standard letter, e.g. $W^{k,p}(\mathcal{O})$. For $p=2$ we denote $\mathbb{W}^{k,2}(\mathcal{O}) = \mathbb{H}^k$ and its norm are denoted by $\|\mathbf{u}\|_k$. By $\mathbb{H}^1_0$ we mean the space of functions in $\mathbb{H}^1$ that vanish on the boundary on $\mathcal{O}$; $\mathbb{H}^1_0$ is a Hilbert space when endowed with the scalar product induced by that of $\mathbb{H}^1$. The usual scalar product on $\mathbb{L}^2$ is denoted by $\langle u, v\rangle$ for $u, v \in \mathbb{L}^2$. Its associated norm is $\|u\|$, $u \in \mathbb{L}^2$. We also introduce the following spaces

$$\begin{aligned}
\mathcal{V} &= \{\mathbf{u} \in [\mathcal{C}^\infty_c(\mathcal{O},\mathbb{R}^n)] \text{ such that } \nabla\cdot\mathbf{u} = 0\}\\
\mathbb{V} &= \text{ closure of } \mathcal{V} \text{ in } \mathbb{H}^1_0(\mathcal{O})\\
\mathbb{H} &= \text{ closure of } \mathcal{V} \text{ in } \mathbb{L}^2(\mathcal{O}).
\end{aligned}$$

We endow $\mathbb{H}$ with the scalar product and norm of $\mathbb{L}^2$. As usual we equip the space $\mathbb{V}$ with the the scalar product $\langle \nabla\mathbf{u}, \nabla\mathbf{v}\rangle$ which is equivalent to the $\mathbb{H}^1(\mathcal{O})$-scalar product.

Let $\Pi : \mathbb{L}^2 \to \mathbb{H}$ be the Helmholtz-Leray projection from $\mathbb{L}^2$ onto $\mathbb{H}$. We denote by $\mathrm{A}_1 = -\Pi\Delta$ the Stokes operator with domain $D(\mathrm{A}_1)$.

From [30, Proposition 1.24] we can define a self-adjoint operator $A : \mathbb{H}^1 \to (\mathbb{H}^1)^*$ by

$$\langle A\mathbf{u}, \mathbf{w}\rangle = a(\mathbf{u},\mathbf{w}) = \int_{\mathcal{O}} \nabla\mathbf{u}\, \nabla\mathbf{w}\, dx, \quad \mathbf{u},\ \mathbf{w} \in \mathbb{H}^1. \tag{2.1}$$

The Neumann Laplacian acting on $\mathbb{R}^n$-valued function will be denoted by A_2, that is,

$$\begin{aligned}
D(\mathrm{A}_2) &:= \left\{\mathbf{u} \in \mathbb{H}^2 : \frac{\partial\mathbf{u}}{\partial\mathbf{n}} = 0 \text{ on } \partial\mathcal{O}\right\},\\
\mathrm{A}_2\mathbf{u} &:= -\sum_{i=1}^n \frac{\partial^2\mathbf{u}}{\partial x_i^2}, \quad \mathbf{u} \in D(\mathrm{A}_2).
\end{aligned} \tag{2.2}$$

It can be shown, see e.g. [17, Theorem 5.31], that $\hat{\mathrm{A}}_2 = I + \mathrm{A}_2$ is a definite positive and self-adjoint operator in the Hilbert space $\mathbb{L}^2 = \mathbb{L}^2(\mathcal{O})$ with compact resolvent. In particular, there exists an ONB $(\phi_k)_{k=1}^\infty$ of $\mathbb{L}^2$ and an a increasing sequence $(\lambda_k)_{k=1}^\infty$ with $\lambda_1 = 0$ and $\lambda_k \nearrow \infty$ as $k \nearrow \infty$ (the eigenvalues of the Neumann Laplacian A_2) such that $\mathrm{A}_2\phi_j = \lambda_j\phi_j$ for any $i \in \mathbb{N}$. For any $\alpha \in [-\frac{1}{2},\infty)$ we denote by $\mathbb{X}_\alpha = D(\hat{\mathrm{A}}_2^{\frac{1}{2}+\alpha})$, the domain of the fractional power operator $\hat{\mathrm{A}}_2^{\frac{1}{2}+\alpha}$. We have the following characteization of the spaces $\mathbb{X}_\alpha$,

$$\mathbb{X}_\alpha = \{\mathbf{u} = \sum_{k\in\mathbb{N}} u_k\phi_k : \sum_k (1+\lambda_k)^{1+2\alpha}|u_k|^2 < \infty\}. \tag{2.3}$$

It can be shown that $\mathbb{X}_\alpha \subset \mathbb{H}^{1+2\alpha}$, for all $\alpha \geq 0$ and $\mathbb{X} := \mathbb{X}_0 = \mathbb{H}^1$.

Similarly, for $\beta \in [0,\infty)$, we denote by $\mathbb{V}^\beta$ the Hilbert space $D(\mathrm{A}_1^\beta)$ endowed with the graph inner product. The Hilbert space $\mathbb{V}^\beta = D(\mathrm{A}_1^\beta)$ for $\beta \in (-\infty,0)$ can be defined by standard extrapolation methods. In particular, the space $\mathbb{V}^{-\beta}$ is the dual of $\mathbb{V}^\beta$ for $\beta \geq 0$. Moreover, for every $\beta,\delta \in \mathbb{R}$ the map A_1^δ is a linear isomorphism between V^β and $V^{\beta-\delta}$.

Throughout this paper $\mathbf{B}^*$ denotes the dual space of a Banach space $\mathbf{B}$. We denote by $\langle\Psi,\mathbf{b}\rangle$ the value of $\Psi \in \mathbf{B}^*$ on $\mathbf{b} \in \mathbf{B}$.

Hereafter we denote by $\|\cdot\|_k$ the norm in the Sobolev, vector or scalar valued, space $H^{k,2}$. We also put

$$H = \mathbb{H} \times \mathbb{X}_0, \;\; V = \mathbb{V} \times D(\mathrm{A}_2) \text{ and } E = D(\mathrm{A}_1) \times \mathbb{X}_1. \tag{2.4}$$

The operator $-\mathrm{A}_2$ is the generator of a C_0 analytic semigroup $\{\mathbb{T}(t)\}_{t\geq 0}$ on $\mathbb{L}^2$ satisfying

$$\mathbb{T}(t)\mathbf{u} = \sum_{k\in\mathbb{N}} e^{-\lambda_k t} u_k \phi_k, \quad \mathbf{u} = \sum_{k\in\mathbb{N}} u_k\phi_k \in \mathbb{L}^2. \tag{2.5}$$

By using the representation (2.3) we can show without any difficulty that the space $\mathbb{X}_0$ is invariant with respect to this semigroup and the restriction of the latter to the former is also a C_0 and analytic semigroup which will be denoted in the sequel by $\{\mathbb{S}_2(t)\}_{t\geq 0}$. The minus infinitesimal generator $\tilde{\mathrm{A}}_2$ of $\{\mathbb{S}_2(t)\}_{t\geq 0}$ is the part of A_2 on $\mathbb{X}_0$, that is,

$$D(\tilde{\mathrm{A}}_2) = \{\mathbf{u} \in D(\mathrm{A}_2) : \mathrm{A}_2\mathbf{u} \in \mathbb{X}_0\}, \;\; \tilde{\mathrm{A}}_2\mathbf{u} = \mathrm{A}_2\mathbf{u} \text{ for any } \mathbf{u} \in D(\tilde{\mathrm{A}}_2).$$

Note that $\mathbb{X}_1 \subset D(\tilde{\mathrm{A}}_2)$.

Next we denote by $\{\mathbb{S}_1(t)\}_{t\geq 0}$ the analytic semigroup generated by $-\mathrm{A}_1$ on $\mathbb{H}$ where A_1 is the Stokes operator.

We also introduce a trilinear form

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = \sum_{i,j=1}^{n} \int_{\mathcal{O}} \mathbf{u}^i \frac{\partial \mathbf{v}^j}{\partial x_i} \mathbf{w}^j dx, \;\; \mathbf{u} \in \mathbb{L}^p, \mathbf{v} \in \mathbb{W}^{1,q}, \text{ and } \mathbf{w} \in \mathbb{L}^r,$$

with numbers $p, q, r \in [1,\infty]$ satisfying

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1.$$

The map b is the trilinear form used in the mathematical analysis of the Navier-Stokes equations, see for instance [35]. It is well known that one can define a bilinear map B_2 defined on $\mathbb{H}^1 \times \mathbb{H}^1$ with values in $(\mathbb{H}^1)^*$ such that

$$\langle B_2(\mathbf{u},\mathbf{v}),\mathbf{w}\rangle = b(\mathbf{u},\mathbf{v},\mathbf{w}) \;\; \text{for any } \mathbf{u}, \;\mathbf{v}, \;\mathbf{w} \in \mathbb{H}^1.$$

We can also a define bilinear map[1] B_1 from $\mathbb{V} \times \mathbb{V}$ with values in $\mathbb{V}^*$ such that

$$\langle B_1(\mathbf{u},\mathbf{v}),\mathbf{w}\rangle = b(\mathbf{u},\mathbf{v},\mathbf{w}) \text{ for } \mathbf{w} \in \mathbb{V}, \text{ and } \mathbf{u},\mathbf{v} \in \mathbb{H}^1.$$

For any $\mathbf{f}, \mathbf{g} \in \mathbb{X}_{\frac{1}{2}} \cap \mathbb{X}_1$ we also set

$$M(\mathbf{f},\mathbf{g}) = \Pi[\nabla \cdot (\nabla\mathbf{f} \otimes \nabla\mathbf{g})].$$

This definition makes sense because $\nabla \cdot (\nabla\mathbf{f} \otimes \nabla\mathbf{g}) \in \mathbb{L}^2$ for any $\mathbf{f},\mathbf{g} \in \mathbb{X}_{\frac{1}{2}} \cap \mathbb{X}_1$.

Let h be an element of $\mathbb{L}^\infty \cap \mathbb{W}^{1,3}$. We define a linear operator G from $\mathbb{L}^2$ into itself by

$$G(\mathbf{d}) = \mathbf{d} \times h.$$

It is straightforward to check that G is bounded and satisfies

$$\|G(\mathbf{d})\| \leq \|h\|_{\mathbb{L}^\infty}\|\mathbf{d}\|.$$

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space equipped with a filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ satisfying the usual condition. Let $W_2 = (W_2(t))_{t\geq 0}$ be a real-valued Wiener process on $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$. Let us assume also that K_1 is a separable Hilbert space and $W_1 = (W_1(t))_{t\geq 0}$ be a K_1-cylindrical Wiener process on $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$. Throughout we assume that W_2 and W_1 are mutually independent. Thus we can assume that $W = (W_1(t), W_2(t))$ is K-cylindrical Wiener process on $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$, where

$$\mathrm{K} = \mathrm{K}_1 \otimes \mathrm{K}_2, \;\; \mathrm{K}_2 = \mathbb{R}.$$

[1]In the context of the Navier-Stokes Equations, the map B_1 is usually denoted by B.

We have the following relation between Stratonovich and Itô's integrals

$$G(\mathbf{d}) \circ dW_2 = \frac{1}{2} G^2(\mathbf{d})\, dt + G(\mathbf{d})\, dW_2,$$

where $G^2 = G \circ G$ and defined by

$$G^2(\mathbf{d}) = G \circ G(\mathbf{d}) = (\mathbf{d} \times h) \times h, \text{ for any } \mathbf{d} \in \mathbb{L}^2.$$

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a function defined by

$$f(\mathbf{d}) = 1_{B(0,1)}(\mathbf{d})(|\mathbf{d}|^2 - 1)\mathbf{d}, \quad \mathbf{d} \in \mathbb{R}^n. \tag{2.6}$$

Remark 2.1. Let f be defined by (2.6). Then there exist positive constants $c > 0$ and $\tilde{c} > 0$ such that

$$|f''(\mathbf{d})| \leq c \text{ and } |f'(\mathbf{d})| \leq \tilde{c} \text{ for any } \mathbf{d}.$$

Now, by performing elementary calculation we can check that

$$\begin{aligned} \|\Delta \mathbf{d}\|^2 =& \|\Delta \mathbf{d} - f(\mathbf{d}) + f(\mathbf{d})\|^2 \leq 2\|\Delta \mathbf{d} - f(\mathbf{d})\|^2 + 2\|f(\mathbf{d})\|^2, \\ \leq& 2\|\Delta \mathbf{d} - f(\mathbf{d})\|^2 + 2\tilde{c}\|\mathbf{d}\|^2, \quad \text{for any } \mathbf{d} \in D(\mathbf{A}_1). \end{aligned}$$

Hence there exists a constant $C > 0$ such that

$$\|\mathbf{d}\|_2^2 \leq C(\|\Delta \mathbf{d} - f(\mathbf{d})\|^2 + 2\tilde{c}\|\mathbf{d}\|^2), \text{ for any } \mathbf{d} \in \mathbb{H}^2(\mathcal{O}). \tag{2.7}$$

With all the above notation, the stochastic equations for nematic liquid crystal (1.9-1.12) can be rewritten as the following stochastic evolution equation in the space H,

$$d\mathbf{y}(t) + \mathbf{A}\mathbf{y}(t)dt + \mathbf{F}(\mathbf{y}(t))dt + \mathbf{L}(\mathbf{y}(t))dt = \mathbf{G}(\mathbf{y}(t))dW(t), \tag{2.8}$$

where, for $\mathbf{y} = (\mathbf{v}, \mathbf{d}) \in E$ and $k = (k_1, k_2) \in \mathrm{K}$,

$$\mathbf{A}\mathbf{y} = \begin{pmatrix} \mathrm{A}_1 & 0 \\ 0 & \mathrm{A}_2 \end{pmatrix} \begin{pmatrix} \mathbf{v} \\ \mathbf{d} \end{pmatrix}, \quad \mathbf{F}(\mathbf{y}) = \begin{pmatrix} B_1(\mathbf{v}, \mathbf{v}) + M(\mathbf{d}) \\ B_2(\mathbf{v}, \mathbf{d}) + f(\mathbf{d}) \end{pmatrix}, \tag{2.9}$$

$$\mathbf{L}(\mathbf{y}) = \begin{pmatrix} 0 \\ -\frac{1}{2}G^2(\mathbf{d}) \end{pmatrix}, \mathbf{G}(\mathbf{y})k = \begin{pmatrix} S(\mathbf{u})k_1 \\ G(\mathbf{d})k_2 \end{pmatrix}. \tag{2.10}$$

Below we will also use the C_0 analytic semigroup $\{\mathbb{S}(t)\}_{t \geq 0}$ on $H = \mathbb{H} \times \mathbb{X}_0$ defined by

$$\mathbb{S}(t) \begin{pmatrix} \mathbf{v} \\ \mathbf{d} \end{pmatrix} = \begin{pmatrix} \mathbb{S}_1(t)\mathbf{v} \\ \mathbb{S}_2(t)\mathbf{d} \end{pmatrix}, \quad (\mathbf{v}, \mathbf{d}) \in H.$$

Its infinitesimal generator is $-\mathbf{A}$, where $\mathbf{A}$ is defined in (2.9). Some properties of $\{\mathbb{S}(t) : t \geq 0\}$ will be given in Lemmata A.3-A.5.

Given two Hilbert spaces K and H, we denote by $\mathcal{J}_2(K, H)$ the Hilbert space of all Hilbert-Schmidt operators from K to H.

The function S is defined in the next set of hypotheses.

Assumption 2.1. *Let $h \in \mathbb{W}^{2,4}$ (hence $h \in \mathbb{W}^{1,3} \cap \mathbb{L}^\infty$) with $h_{|\partial\mathcal{O}} = 0$.*

We assume that $S : \mathbb{H} \to \mathcal{J}_2(\mathrm{K}_1, \mathbb{V})$ is a globally Lipschitz map. In particular, there exists $\ell_5 \geq 0$ such that

$$\|S(\mathbf{u})\|^2_{\mathcal{J}_2(\mathrm{K}_1, \mathbb{V})} \leq \ell_5(1 + \|\mathbf{u}\|^2), \quad \textit{for any } \mathbf{u} \in \mathbb{H}.$$

Let us recall the following notations/definition which are borrowed from [3] or [22].

Definition 2.2. For a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with given right-continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$, a stopping time τ is called accessible iff there exists an increasing sequence of stopping times τ_n such that a.s. $\tau_n < \tau$ and $\lim_{n \to \infty} \tau_n = \tau$, see [22].

Notation. For a stopping time τ we set

$$\begin{aligned} \Omega_t(\tau) &= \{\omega \in \Omega : t < \tau(\omega)\}, \\ [0, \tau) \times \Omega &= \{(t, \omega) \in [0, \infty) \times \Omega : 0 \leq t < \tau(\omega)\}. \end{aligned}$$

Definition 2.3. A process $\eta : [0,\tau) \times \Omega \to X$ (we will also write $\eta(t)$, $t < \tau$), where X is a metric space, is admissible iff

(i) it is adapted, i.e. $\eta|_{\Omega_t} : \Omega_t \to X$ is $\mathcal{F}_t$ measurable, for any $t \geq 0$;
(ii) for almost all $\omega \in \Omega$, the function $[0,\tau(\omega)) \ni t \mapsto \eta(t,\omega) \in X$ is continuous.

A process $\eta : [0,\tau) \times \Omega \to X$ is progressively measurable iff, for any $t > 0$, the map

$$[0, t\wedge\tau)\times\Omega \ni (s,\omega) \mapsto \eta(s,\omega) \in X$$

is $\mathcal{B}_{t\wedge\tau} \times \mathcal{F}_{t\wedge\tau}$ measurable.
Two processes $\eta_i : [0,\tau_i)\times\Omega \to X$, $i = 1,2$ are called equivalent (we will write $(\eta_1,\tau_1) \sim (\eta_2,\tau_2)$) iff $\tau_1 = \tau_2$ a.s. and for any $t > 0$ the following holds

$$\eta_1(\cdot,\omega) = \eta_2(\cdot,\omega) \text{ on } [0,t]$$

for a.a. $\omega \in \Omega_t(\tau_1) \cap \Omega_t(\tau_2)$.
Note that if processes $\eta_i : [0,\tau_i)\times\Omega \to X$, $i = 1,2$ are admissible and for any $t > 0$ $\eta_1(t)|_{\Omega_t(\tau_1)} = \eta_2(t)|_{\Omega_t(\tau_2)}$ a.s. then they are also equivalent.

We now define some concepts of solution to Eq. (3.25), see [7, Def. 4.2] or [28, Def. 2.1].

Definition 2.4. Assume that a V-valued $\mathcal{F}_0$ measurable random variable $\mathbf{y}_0$ with $\mathbb{E}\|\mathbf{y}_0\|^2 < \infty$ is given. A local mild solution to problem (3.10) (with the initial time 0) is a pair $(\mathbf{y},\tau)$ such that

(1) τ is an accessible $\mathbb{F}$-stopping time,
(2) $\mathbf{y} : [0,\tau) \times \Omega \to V$ is an admissible process,
(3) there exists an approximating sequence $(\tau_m)_{m\in\mathbb{N}}$ of $\mathbb{F}$ finite stopping times such that $\tau_m \nearrow \tau$ a.s. and, for every $m \in \mathbb{N}$ and $t \geq 0$, we have

$$\mathbb{E}\Big(\sup_{s\in[0,t\wedge\tau_m]} \|\mathbf{y}(s)\|^2 + \int_0^{t\wedge\tau_m} |\mathbf{y}(s)|_E^2\, ds\Big) < \infty, \tag{2.11}$$

$$\begin{aligned}\mathbf{y}(t\wedge\tau_m) &= \mathbb{S}(t\wedge\tau_m)\mathbf{y}_0 - \int_0^{t\wedge\tau_m} \mathbb{S}(t\wedge\tau_m - s)[\mathbf{F}(\mathbf{y}(s)) + \mathbf{L}(\mathbf{y}(s)]ds \\ &+ \int_0^\infty \mathbb{1}_{[0,t\wedge\tau_m)}\mathbb{S}(t\wedge\tau_m - s)\mathbf{G}(\mathbf{y}(s))\, d\mathbb{W}(s).\end{aligned} \tag{2.12}$$

Along the lines of the paper [3], we say that a local solution $\mathbf{y}(t)$, $t < \tau$ is global iff $\tau = \infty$ a.s.

We will check that $\mathbf{F}$ satisfies the assumption of Proposition 3.12 with $H = \mathbb{H} \times \mathbb{X}_0$, $V = \mathbb{V} \times D(A_1)$ and $E = D(\mathrm{A}_1) \times \mathbb{X}_1$. For this purpose we will prove several estimates.

Lemma 2.5. *There exist some positive constants c_1 and c_2 such that for any $(\mathbf{v}_i,\mathbf{d}_i) \in E$, $i= 1,2$ we have, with $a = \frac{n}{4}$,*

$$\begin{aligned}\|B_1(\mathbf{v}_1,\mathbf{v}_1) - B_1(\mathbf{v}_2,\mathbf{v}_2)\| \leq c_1\Big(&\|\mathrm{A}_1^{\frac12}(\mathbf{v}_1-\mathbf{v}_2)\|\|\mathrm{A}_1^{\frac12}\mathbf{v}_1\|^{1-a}\|\mathrm{A}_1\mathbf{v}_1\|^a \\ &+ \|\mathrm{A}_1^{\frac12}(\mathbf{v}_1-\mathbf{v}_2)\|^{1-a}\|\mathrm{A}_1(\mathbf{v}_1-\mathbf{v}_2)\|^a\|\mathrm{A}_1^{\frac12}(\mathbf{v}_2)\|\Big)\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}\|M(\mathbf{d}_1) - M(\mathbf{d}_2)\| \leq c_2\Big(&\|\mathbf{d}_1-\mathbf{d}_2\|_2\|\mathbf{d}_1\|_2^{1-a}\|\mathbf{d}_1\|_3^a \\ &+ \|\mathbf{d}_1-\mathbf{d}_2\|_2^{1-a}\|\mathbf{d}_1-\mathbf{d}_2\|_3^a\|\mathbf{d}_2\|_2\Big),\end{aligned} \tag{2.14}$$

Proof. Set $(\mathbf{w},\bar{\mathbf{d}}) = (\mathbf{v}_1-\mathbf{v}_2, \mathbf{d}_1-\mathbf{d}_2)$. We start with the proof of the estimate (2.13). Notice that the left-hand-side of (2.13) is equal to

$$\|B_1(\mathbf{w},\mathbf{v}_1) + B_1(\mathbf{v}_2,\mathbf{w})\|.$$

Now we estimate the last identity. We have

$$\|B_1(\mathbf{w},\mathbf{v}_1)+B_1(\mathbf{v}_2,\mathbf{w})\|\le C\|\mathbf{w}\|_{\mathbb{L}^4}\|\nabla\mathbf{v}_1\|_{\mathbb{L}^4}+\|\mathbf{v}_2\|_{\mathbb{L}^4}\|\nabla\mathbf{w}\|_{\mathbb{L}^4},$$

from which along with and Eq. (A.1) and the embedding (A.3) we easily derive the estimate (2.13).

Next we show that (2.14) holds. From elementary calculi we infer the existence of a constant $C>0$ such that

$$\|M(\mathbf{f},\mathbf{g})\|\le C\|D^2\mathbf{f}\|\|\nabla\mathbf{g}\|_{\mathbb{L}^\infty}+\|\nabla\mathbf{f}\|_{\mathbb{L}^4}\|D^2\mathbf{g}\|_{\mathbb{L}^4}.$$

Owing to the embedding (A.3) it is not difficult to check that

$$\|M(\mathbf{f},\mathbf{g})\|\le C\|\mathbf{f}\|_2\Big(\|\nabla\mathbf{g}\|_{\mathbb{L}^\infty}+\|D^2\mathbf{g}\|_{\mathbb{L}^4}\Big).$$

Owing to Eq. (A.1), Eq. (A.4) and the embedding (A.3) we obtain that

$$\|M(\mathbf{f},\mathbf{g})\|\le C\|\mathbf{f}\|_2\|\mathbf{g}\|_2^{1-a}\|\mathbf{g}\|_3^a,\quad a=\frac{n}{4}. \tag{2.15}$$

Note that

$$M(\mathbf{d}_1)-M(\mathbf{d}_2)=M(\mathbf{d}_1-\mathbf{d}_2,\mathbf{d}_1)+M(\mathbf{d}_2,\mathbf{d}_1-\mathbf{d}_2).$$

From this last identity and Eq. (2.15) we easily deduce the inequality (2.14). This ends the proof of Lemma 2.5. □

Lemma 2.6. *There exist a constant $c_3>0$ such that for any $(\mathbf{v}_i,\mathbf{d}_i)\in E$, $i=1,2$ we have*

$$\begin{aligned}\|B_2(\mathbf{v}_1,\mathbf{d}_1)-B_2(\mathbf{v}_2,\mathbf{d}_2)\|_1\le c_3\Big(&\|\mathrm{A}_1^{\frac12}(\mathbf{v}_1-\mathbf{v}_2)\|\|\mathbf{d}_1\|_2^{1-a}\|\mathbf{d}_1\|_3^a\\ &+\|(\mathbf{d}_1-\mathbf{d}_2)\|_2^{1-a}\|(\mathbf{d}_1-\mathbf{d}_2)\|_3^a\|\mathrm{A}_1^{\frac12}(\mathbf{v}_2)\|\Big)\end{aligned} \tag{2.16}$$

Proof. As in the proof of Lemma 2.5 we set $(\mathbf{w},\bar{\mathbf{d}})=(\mathbf{v}_1-\mathbf{v}_2,\mathbf{d}_1-\mathbf{d}_2)$ and notice that the left hand side of Eq. (2.16) is equal to

$$B_2(\mathbf{w},\mathbf{d}_1)+B_2(\mathbf{d}_2,\mathbf{w}):=J_1+J_2.$$

Now we want to estimate $\|J_i\|_1=\sqrt{\|J_i\|^2+\|\nabla J_i\|^2}$ for $i=1,2$. Since estimating $\|J_i\|$ is easy we will just focus on the term $\|\nabla J_i\|$. There exists a constant $C>0$ such that

$$\begin{aligned}\|\nabla J_1\|&\le C\Big(\|\nabla\mathbf{w}\nabla\mathbf{d}_1\|+\|D^2\mathbf{d}_1\mathbf{w}\|\Big),\\ &\le C\Big(\|\nabla\mathbf{w}\|\|\nabla\mathbf{d}_1\|_{\mathbb{L}^\infty}+\|\mathbf{w}\|_{\mathbb{L}^4}\|D^2\mathbf{d}_1\|_{\mathbb{L}^4}\Big).\end{aligned}$$

Invoking Eq. (A.1), Eq. (A.2) and the embedding (A.3) we infer that with $a=\frac{n}{4}$,

$$\|\nabla J_1\|\le C\|\mathrm{A}_1^{\frac12}\mathbf{w}\|\Big(\|\nabla\mathbf{d}_1\|_1^{1-a}\|D^2\mathbf{d}_1\|_1^a+\|D^2\mathbf{d}_1\|^{1-a}\|D^3\mathbf{d}_1\|^a\Big).$$

This last inequality implies that there exits $\tilde{c}>0$ such that with $a=\frac{n}{4}$,

$$\|\nabla J_1\|\le\tilde{c}\|\mathrm{A}_1^{\frac12}\mathbf{w}\|\|\mathbf{d}_1\|_2^{1-a}\|\mathbf{d}_1\|_3^a.$$

Using similar argument we can also prove that (again with $a=\frac{n}{4}$)

$$\|\nabla J_2\|\le\tilde{c}\|\mathrm{A}_1^{\frac12}\mathbf{v}_2\|\|\mathbf{d}_1-\mathbf{d}_2\|_2^{1-a}\|\mathbf{d}_1-\mathbf{d}_2\|_3^a.$$

The inequality (2.16) easily follows from these last two estimates. □

Lemma 2.7. *There exists $c_4 > 0$ such that for any $\mathbf{d}_i \in \mathbb{X}_{\frac{1}{2}} \cap \mathbb{X}_1$ with $i = 1, 2$.*

$$\begin{aligned} \|f(\mathbf{d}_1) - f(\mathbf{d}_2)\|_1 \leq c_4 \Big(& \|\mathbf{d}_1 - \mathbf{d}_2\|_2 \|\mathbf{d}_1\|_2^{1-a} \|\mathbf{d}_1\|_3^a \\ & + \|\mathbf{d}_1 - \mathbf{d}_2\|_2^{1-a} \|\mathbf{d}_1 - \mathbf{d}_2\|_3^a \|\mathbf{d}_2\|_2 + \|\mathbf{d}_1 - \mathbf{d}_2\|_2 \Big). \end{aligned} \tag{2.17}$$

Proof. As in the proof of Lemma 2.6 we will just estimate $\|\nabla (f(\mathbf{d}_1) - f(\mathbf{d}_2))\|$. Again we set $\bar{\mathbf{d}} = \mathbf{d}_1 - \mathbf{d}_2$. There exists $C > 0$ such that

$$\begin{aligned} \|\nabla (f(\mathbf{d}_1) - f(\mathbf{d}_2))\| &= \|\nabla \mathbf{d}_1 f'(\mathbf{d}_1) - \nabla \mathbf{d}_2 f'(\mathbf{d}_2)\|, \\ &\leq \|\nabla \bar{\mathbf{d}}\ f'(\mathbf{d}_1)\| + \|\nabla \mathbf{d}_2 \left(f'(\mathbf{d}_1) - f'(\mathbf{d}_2)\right)\|, \\ &\leq C \Big(\|\nabla \bar{\mathbf{d}}\| \|f'(\mathbf{d}_1)\|_{\mathbb{L}^\infty} + \|\nabla \mathbf{d}_2\| \|f'(\mathbf{d}_1) - f'(\mathbf{d}_2)\|_{\mathbb{L}^\infty} \Big). \end{aligned}$$

Owing to the definition of f we derive from the last line of the above inequalities that

$$\|\nabla (f(\mathbf{d}_1) - f(\mathbf{d}_2))\| \leq C \Big(\|\nabla \bar{\mathbf{d}}\| \|\mathbf{d}_1\|_{\mathbb{L}^\infty} + \|\nabla \mathbf{d}_2\| \|\bar{\mathbf{d}}\|_{\mathbb{L}^\infty} + \|\nabla \bar{\mathbf{d}}\| \Big).$$

Plugging Eq. (A.2) in this estimate yields, with $a = \frac{n}{4}$,

$$\|\nabla (f(\mathbf{d}_1) - f(\mathbf{d}_2))\| \leq C \Big(\|\nabla \bar{\mathbf{d}}\| \|\mathbf{d}_1\|_{\mathbb{L}^4}^{1-a} \|\nabla \mathbf{d}_1\|_{\mathbb{L}^4}^a + \|\nabla \mathbf{d}_2\| \|\bar{\mathbf{d}}\|_{\mathbb{L}^4}^{1-a} \|\nabla \bar{\mathbf{d}}\|_{\mathbb{L}^4}^a + \|\nabla \bar{\mathbf{d}}\| \Big).$$

Thanks to the continuous embedding $\mathbb{H}^{k+1} \subset \mathbb{H}^1 \subset \mathbb{L}^4$ with $k \geq 0$ we derive easily from the above inequality that, again with $a = \frac{n}{4}$,

$$\|\nabla (f(\mathbf{d}_1) - f(\mathbf{d}_2))\| \leq C \Big(\|\bar{\mathbf{d}}\|_2 \|\mathbf{d}_1\|_2^{1-a} \|\nabla \mathbf{d}_1\|_3^a + \|\mathbf{d}_2\|_2 \|\bar{\mathbf{d}}\|_2^{1-a} \|\bar{\mathbf{d}}\|_3^a + \|\nabla \bar{\mathbf{d}}\| \Big).$$

□

For two Banach spaces $(B_i, \|\cdot\|_{B_i})$ with $i = 1, 2$ we endow the product space $B_1 \times B_2$ with the norm

$$|(b_1, b_2)| = \sqrt{\|b_1\|_{B_1}^2 + \|b_2\|_{B_2}^2}.$$

Proposition 2.8. *There exists a certain constant $C_0 > 0$ such that for any $\mathbf{y}_i = (\mathbf{v}_i, \mathbf{d}_i)$, $i = 1, 2$, with $\alpha = \frac{n}{4}$, we have*

$$\|\mathbf{F}(\mathbf{y}_1) - \mathbf{F}(\mathbf{y}_2)\|_H \leq C_0 \|\mathbf{y}_1 - \mathbf{y}_2\|_V \Big[\|\mathbf{y}_1\|_V^{1-\alpha} \|\mathbf{y}_1\|_E^\alpha + \|\mathbf{y}_1 - \mathbf{y}_2\|_E^\alpha \|\mathbf{y}_1 - \mathbf{y}_2\|_V^{-\alpha} \|\mathbf{y}_2\|_V + 1 \Big]. \tag{2.18}$$

Proof. The proposition is a consequence of Lemma 2.5, Lemma 2.6 and Lemma 2.7. Its proof is easy and we omit it. □

For any integer $k > 1$ let

$$\tau_k = \inf\{t \geq 0 : \|\mathrm{A}_1^{\frac{1}{2}} \mathbf{v}(t)\| + \|\Delta \mathbf{d}(t)\| > k\},$$

and $\tau_\infty = \lim_{k\to\infty} \tau_k$. Hereafter, we set $t_k = t \wedge \tau_k$ for any $t > 0$ and for a vector-valued function $\mathbf{u} : [0, t_k] \to \mathbf{B}$ we will write $\int_0^{t_k} \mathbf{u}\, ds := \int_0^{t_k} \mathbf{u}(s) ds$ for any $t > 0$.
Our first main result is contained in the following theorem. It is basically a corollary of a general theorem that we will state and prove in the next section.

Theorem 2.9. *Let $n = 2, 3$ and $(\mathbf{v}_0, \mathbf{d}_0) \in D(\mathrm{A}_1^{\frac{1}{2}}) \times \mathbb{X}_{\frac{1}{2}}$. The stochastic equation (2.8) for the liquid crystals admits a local-maximal strong solution $(\mathbf{y}, \tau_\infty)$ provided that Assumption 2.1 holds.*

Proof. Lemma A.3-A.5 show that $\{\mathbb{S}(t)\}_{t\geq 0}$ on $H = \mathbb{H} \times \mathbb{X}_0$ satisfies Assumption 3.3. Thanks to Proposition 2.8 we can apply the Theorem 3.15 and Theorem 3.16 to deduce the existence of local and maximal strong solution to problem (2.8). This concludes the proof of the theorem. □

The second result is about global solvability of the stochastic equation for two dimensional nematic liquid crystal.

Theorem 2.10. *Assume that $n = 2$ and $\mathbf{v}_0 \in D(\mathrm{A}_1^{\frac{1}{2}})$, and $\mathbf{d}_0 \in \mathbb{X}_{\frac{1}{2}}$. Then the stochastic equation (2.8) for nematic liquid crystals has a global strong solution provided Assumption 2.1 hold.*

Proof. For any $\alpha > 0$ and $p, q \geq 1$ with $p^{-1} + q^{-1} = 1$ let $C(\alpha, p, q)$ be the constant from the Young inequality

$$ab \leq C(\alpha, p, q)a^p + \alpha b^q.$$

Let us choose $p = \frac{8}{n+4}$, $q = \frac{8}{4-n}$, and $\alpha = 1$. Let us set $\Phi(s) = e^{-\int_0^s \phi(r)dr}$, where the function ϕ is defined by

$$\phi(s) = C(1, p, q)\|\mathbf{v}(s)\|^2\|\mathrm{A}_1^{\frac{1}{2}}\mathbf{v}(s)\|^{\frac{2n}{4-n}}.$$

For $\mathbf{d} \in D(A)$ let us set

$$\Psi(\mathbf{d}) = \|-A\mathbf{d} - f(\mathbf{d})\|^2.$$

By arguing as in [6, pp. 123] we have

$$\begin{aligned}
\mathbb{P}(\tau_k < t) &\leq \mathbb{E}\Big(1_{\{\tau_k<t\}}e^{-\frac{1}{2}\Phi(t_k)}(\|\mathrm{A}_1^{\frac{1}{2}}\mathbf{v}(t_k)\| + \sqrt{\Psi(\mathbf{d}(t_k))})e^{\frac{1}{2}\int_0^{t_k}\phi(r)dr}\Big),\\
&\leq \mathbb{E}\Big(1_{\{\tau_k<t\}}e^{-\frac{1}{2}\Phi(t_k)}(\|\mathrm{A}_1^{\frac{1}{2}}\mathbf{v}(t_k)\| + \sqrt{\Psi(\mathbf{d}(t_k))})\Big)\\
&\quad + \mathbb{P}\Big(\int_0^{t_k}\|\mathbf{v}\|^2\|\mathrm{A}_1^{\frac{1}{2}}\mathbf{v}\|^2 ds > \frac{\log k}{2C(1,p,q)}\Big),\\
&\leq \frac{1}{k}\mathbb{E}\Big(\Phi(t_k)(\|\mathrm{A}_1^{\frac{1}{2}}\mathbf{v}(t_k)\|^2 + \Psi(\mathbf{d}(t_k)))\Big) + \frac{2C(1,p,q)}{\log k}\mathbb{E}\int_0^{t_k}\|\mathbf{v}\|^2\|\mathrm{A}_1^{\frac{1}{2}}\mathbf{v}\|^2 ds.
\end{aligned}$$

Thanks to Proposition B.1, Remark 2.1, Eq. (B.3) and Eq. (2.7),

$$\mathbb{P}(\tau_k < t) \leq \frac{1}{k}[C + C(\mathbf{v}_0, \mathbf{d}_0)e^{C(h)t}] + \frac{2C(1,p,q)}{\log k}\mathbb{E}\int_0^{t_k}\|\mathbf{v}\|^2\|\mathrm{A}_1^{\frac{1}{2}}\mathbf{v}\|^2 ds.$$

But from Proposition B.1 the solution $(\mathbf{v}, \mathbf{d})$ satisfies

$$\sup_{0\leq s\leq t_k}\mathbb{E}\int_0^{t_k}\|\mathbf{v}(s)\|^2\|\mathrm{A}_1^{\frac{1}{2}}\mathbf{v}(s)\|^2 ds < C(\mathbf{v}_0, \mathbf{d}_0)e^{C(h,4)t}.$$

Hence, combining this latter equation with the former one implies that

$$\lim_{k\to\infty}\mathbb{P}(\tau_k < t) = 0,$$

from which $\mathbb{P}\Big(\tau_\infty < \infty\Big) = 0$ follows. □

2.2. **Maximum Principle type Theorem.** In this subsection we show that if the initial value $\mathbf{d}_0$ is in the unit ball then so are the values of the vector director $\mathbf{d}$. That is, we must show that $|\mathbf{d}(t)|^2 \leq 1$ almost all $(\omega, t, x) \in \Omega \times [0, T] \times \mathcal{O}$. In fact we have the following proposition.

Proposition 2.11. *Assume that $n \leq 3$ and that a process $(\mathbf{v}, \mathbf{d}) = (\mathbf{v}(t), \mathbf{d}(t))$, $t \in [0, T]$, is a solution to problem (2.8) with initial condition $\mathbf{d}_0$ such that $|\mathbf{d}_0|^2 \leq 1$ for almost all $(\omega, x) \in \Omega \times \mathcal{O}$. Then $|\mathbf{d}(t)|^2 \leq 1$ for almost all $(\omega, t, x) \in \Omega \times [0, T] \times \mathcal{O}$.*

Proof. We follow the idea in [9, Lemma 2.1] and [13, Proof of Theorem 4, Page 513]. Let $\varphi : \mathbb{R} \to [0,1]$ be an increasing function of class C^∞ such that

$$\varphi(s) = 0 \text{ iff } s \in (-\infty, 1],$$
$$\varphi(s) = 1 \text{ iff } s \in [2, +\infty).$$

Let $\{\varphi_m; m \in \mathbb{N}\}$ and $\{\phi_m, m \in \mathbb{N}\}$ be two sequences of smooth function from $\mathbb{R}^n$ defined by

$$\begin{aligned}\varphi_m(\mathbf{d}) =& \varphi(m(|\mathbf{d}|^2 - 1)),\\ \phi_m(\mathbf{d}) =& (|\mathbf{d}|^2 - 1)\varphi_m(\mathbf{d}), \mathbf{d} \in \mathbb{R}^n.\end{aligned}$$

Define a sequence of function $\{\Psi_m, m \in \mathbb{N}\}$ by

$$\begin{aligned}\Psi_m(\mathbf{d}) =& \|\phi_m(\mathbf{d})\|^2,\\ =& \int_{\mathcal{O}} (|\mathbf{d}|^2 - 1)^2 [\varphi_m(\mathbf{d})]^2 dx, \ \mathbf{d} \in \mathbb{L}^4(\mathcal{O}),\end{aligned}$$

for any $m \in \mathbb{N}$. It is clear that $\Psi_m : \mathbb{H}^2 \to \mathbb{R}$ is twice (Fréchet) differentiable and its first and second derivatives satisfy

$$\Psi_m(\mathbf{d})(h) = 4 \int_{\mathcal{O}} \left((|\mathbf{d}|^2 - 1)\varphi_m(\mathbf{d})\mathbf{d} \cdot h\right) dx + 2m \int_{\mathcal{O}} (|\mathbf{d}|^2 - 1)^2 \varphi_m(\mathbf{d})(\mathbf{d} \cdot h) dx,$$

and

$$\begin{aligned}\Psi_m''(\mathbf{d})(k,h) = 8 \int_{\mathcal{O}} \Big[\varphi_m(\mathbf{d})(\mathbf{d} \cdot k)(\mathbf{d} \cdot h)\Big] dx &+ 4 \int_{\mathcal{O}} \left(\varphi_m(\mathbf{d})(|\mathbf{d}|^2 - 1)(k \cdot h)\right) dx\\ &+ 16m \int_{\mathcal{O}} \left((|\mathbf{d}|^2 - 1)\varphi_m(\mathbf{d})(\mathbf{d} \cdot k)(\mathbf{d} \cdot h)\right) dx\\ &+ 4m^2 \int_{\mathcal{O}} \left((|\mathbf{d}|^2 - 1)^2 \varphi_m''(\mathbf{d})(\mathbf{d} \cdot k)(\mathbf{d} \cdot h)\right) dx\\ &+ 2m \int_{\mathcal{O}} (|\mathbf{d}|^2 - 1)^2 \varphi_m(\mathbf{d})(k \cdot h) dx,\end{aligned}$$

for any $\mathbf{d} \in \mathbb{H}^2$ and $h, k \in \mathbb{L}^2(\mathcal{O})$. In particular, for any k, h such that $k \perp \mathbf{d}$ and $h \perp \mathbf{d}$

$$\begin{aligned}\Psi_m(\mathbf{d})(h) =& 0,\\ \Psi_m''(\mathbf{d})(k,h) =& 4 \int_{\mathcal{O}} (|\mathbf{d}|^2 - 1)\varphi_m(\mathbf{d})(k \cdot h) dx + 2m \int_{\mathcal{O}} (|\mathbf{d}|^2 - 1)^2 \varphi_m(\mathbf{d})(k \cdot h) dx.\end{aligned}$$

It follows from Itô's formula (see [31, Theorem I.3.3.2, Page 147]) that

$$d[\Psi_m(\mathbf{d})] = \Psi_m(\mathbf{d}) \left(\Delta \mathbf{d} - B_2(\mathbf{v}, \mathbf{d}) - \frac{1}{\varepsilon^2} f(\mathbf{d}) + \frac{1}{2} G^2(\mathbf{d})\right) dt + \frac{1}{2} \Psi_m''(\mathbf{d})(G(\mathbf{d}), G(\mathbf{d})) dt.$$

The integral stochastic vanishes because $G(\mathbf{d}) \perp \mathbf{d}$. Owing to the identity

$$-|\mathbf{d} \times h|^2 = \mathbf{d} \cdot ((\mathbf{d} \times h) \times h),$$

we have

$$\frac{1}{2} \Psi_m''(G(\mathbf{d}), G(\mathbf{d})) + \frac{1}{2} \Psi_m'(G^2(\mathbf{d})) = 0.$$

Hence

$$d[\Psi_m(\mathbf{d})] = \Psi_m(\mathbf{d}) \left(\Delta \mathbf{d} - B_2(\mathbf{v}, \mathbf{d}) - \frac{1}{\varepsilon^2} f(\mathbf{d})\right) dt \tag{2.19}$$

Noticing that from the definition of φ_m and the Lebesgue Dominated Convergence Theorem we have for $\mathbf{d} \in \mathbb{H}^2, h \in \mathbb{L}^2$

$$\begin{aligned}\lim_{m\to\infty} \Psi_m(\mathbf{d}) =& \| \left(|\mathbf{d}|^2 - 1\right)_+ \|^2,\\ \lim_{m\to\infty} \Psi_m(\mathbf{d})(k) =& 4 \int_{\mathcal{O}} [\left(|\mathbf{d}|^2 - 1\right)_+ \mathbf{d} \cdot h]\, dx.\end{aligned}$$

Hence, we obtain from letting $m \to \infty$ in Eq. (2.19) that for almost all $(\omega, t) \in \Omega \times [0, T]$

$$y(t) - y(0) + 4 \int_0^t \left(\int_{\mathcal{O}} \left[-\Delta \mathbf{d} + (\mathbf{v} \cdot \nabla)\mathbf{d} + \frac{1}{\varepsilon^2} f(\mathbf{d}) \right] \cdot \left[\mathbf{d} \left(|\mathbf{d}|^2 - 1\right)_+ \right] dx \right) ds = 0,$$

where $y(t) = \| \left(|\mathbf{d}(t)|^2 - 1\right)_+ \|^2$. Let us set $\xi = \left(|\mathbf{d}|^2 - 1\right)_+$, it follows from [1, Exercise 7.1.5, p 283] that $\xi \in \mathbb{H}^1$ if $\mathbf{d} \in \mathbb{H}^1$. Thus, since $\frac{\partial \mathbf{d}}{\partial \mathbf{n}} = 0$ we derive from integration-by-parts that

$$-4 \int_0^t \left(\int_{\mathcal{O}} \Delta \mathbf{d} \cdot \mathbf{d} \left(|\mathbf{d}|^2 - 1\right)_+ dx \right) ds = \int_0^t \left(\int_{\mathcal{O}} \left(2\nabla(|\mathbf{d}|^2) \cdot \nabla \xi + 4\xi |\nabla \mathbf{d}|^2 \right) dx \right) ds,$$

Since $\xi \geq 0$ and $|\nabla \mathbf{d}|^2 \geq 0$ a.e. $(t, x) \in \mathcal{O} \times [0, T]$ we easily derive from the above identity that

$$-4 \int_0^t \left(\int_{\mathcal{O}} \Delta \mathbf{d} \cdot \mathbf{d} \left(|\mathbf{d}|^2 - 1\right)_+ dx \right) ds \geq 2 \int_0^t \left(\int_{\mathcal{O}} \nabla(|\mathbf{d}|^2 - 1) \cdot \nabla \xi dx \right) ds.$$

Thanks to [1, Exercise 7.1.5, p 283] we have

$$\int_0^t \left(\int_{\mathcal{O}} \nabla(|\mathbf{d}|^2 - 1) \cdot \nabla \xi dx \right) ds = \int_0^t \int_{\mathcal{O}} |\nabla(|\mathbf{d}|^2 - 1)|^2 \mathbb{1}_{\{|\mathbf{d}|^2 > 1\}} dx\, ds,$$

which implies that

$$-4 \int_0^t \left(\int_{\mathcal{O}} \Delta \mathbf{d} \cdot \mathbf{d} \left(|\mathbf{d}|^2 - 1\right)_+ dx \right) ds \geq \int_0^t \int_{\mathcal{O}} |\nabla(|\mathbf{d}|^2 - 1)|^2 \mathbb{1}_{\{|\mathbf{d}|^2 > 1\}} dx\, ds.$$

We also have

$$\begin{aligned}4 \int_0^t \left(\int_{\mathcal{O}} [(\mathbf{v} \cdot \nabla)\mathbf{d}] \cdot [\mathbf{d} \left(|\mathbf{d}|^2 - 1\right)_+] dx \right) ds =& 2 \int_0^t \left(\int_{\mathcal{O}} [(\mathbf{v} \cdot \nabla)(|\mathbf{d}|^2)][\left(|\mathbf{d}|^2 - 1\right)_+] dx \right) ds,\\ =& \int_0^t \left(\int_{\mathcal{O}} (\mathbf{v} \cdot \nabla)\xi \xi dx \right) ds,\\ =& 0.\end{aligned}$$

Since $f(\mathbf{d}) = 0$ for $|\mathbf{d}|^2 > 1$ and $\xi f(\mathbf{d}) = 0$ for $|\mathbf{d}|^2 \leq 1$ we have

$$4 \int_0^t \left(\int_{\mathcal{O}} \xi f(\mathbf{d}) \cdot \mathbf{d}\, dx \right) ds = 0.$$

Therefore we see that $y(t)$ satisfies the estimate

$$y(t) + 2 \int_0^t \int_{\{|\mathbf{d}|^2 > 1\}} |\nabla(|\mathbf{d}|^2 - 1)_+|^2 ds \leq y(0),$$

for almost all $(\omega, t) \in \Omega \times [0, T]$. Since the second term in the left hand side of the above inequality is positive and $y(0) = \|(|\mathbf{d}_0|^2 - 1)_+\|^2$ and by assumption $|\mathbf{d}_0|^2 \leq 1$ for almost all $(\omega, t, x) \in \Omega \times [0, T] \times \mathcal{O}$ we derive that

$$y(t) = 0,$$

for almost all $(\omega, t) \in \Omega \times [0, T]$, $T \geq 0$. Hence we have $|\mathbf{d}|^2 \leq 1$ a.e. $(\omega, t, x) \in \Omega \times [0, T] \times \mathcal{O}$, $T \geq 0$. $\square$

3. Strong solution for an abstract stochastic equation

The goal of this section is to prove a general result about the existence of local and maximal solution to an abstract stochastic partial differential equations with locally Lipschitz continuous coefficients. This is achieved by using some truncation and fixed point methods.

3.1. **Notations and Preliminary.** Let V, E and H be separable Banach spaces such that $E \subset V$ continuously. We denote the norm in V by $\|\cdot\|$ and we put

$$X_T := C([0,T];V) \cap L^2(0,T;E) \tag{3.1}$$

with the norm

$$|u|^2_{X_T} = \sup_{s\in[0,T]} \|u(s)\|^2 + \int_0^T |u(s)|^2_E \, ds. \tag{3.2}$$

Let F and G be two nonlinear mappings satisfying the following sets of conditions.

Assumption 3.1. *Suppose that $F : E \to H$ is such that $F(0) = 0$ and there exists $p \geq 1$, $\alpha \in [0,1)$ and $C > 0$ such that*

$$|F(y) - F(x)|_H \leq C\Big[\|y - x\| \|y\|^{p-\alpha} |y|^\alpha_E + |y - x|^\alpha_E \|y - x\|^{1-\alpha} \|x\|^p\Big], \tag{3.3}$$

for any $x, y \in \mathbb{E}$.

Assumption 3.2. *Assume that $G : E \to V$ such that $G(0) = 0$ and there exists $k \geq 1$, $\beta \in [0,1)$ and $C_G > 0$ such that*

$$\|G(y) - G(x)\| \leq C_G\Big[\|y - x\| \|y\|^{k-\beta} |y|^\beta_E + |y - x|^\beta_E \|y - x\|^{1-\beta} \|x\|^k\Big], \tag{3.4}$$

for any $x, y \in \mathbb{E}$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ satisfying the usual condition. By $M^2(X_T)$ we denote the space of all progressively measurable E-values processes whose trajectories belong to X_T almost surely endowed with a norm

$$|u|^2_{M^2(X_T)} = \mathbb{E}\Big[\sup_{s\in[0,T]} \|u(s)\|^2 + \int_0^T |u(s)|^2_E \, ds\Big]. \tag{3.5}$$

Let us also formulate the following assumptions.

Assumption 3.3. *Suppose that $E \subset V \subset H$ continuously. Consider (for simplicity) a one-dimensional Wiener process $W(t)$.*
Assume that $S(t)$, $t \in [0,\infty)$, is a family of bounded linear operators on the space H such that there exists two positive constants C_1 and C_2 with the following properties .
(i) For every $T > 0$ and every $f \in L^2(0,T;H)$ a function $u = S * f$ defined by

$$u(t) = \int_0^T S(t - r) f(r) \, dt, \;\; t \in [0,T]$$

belongs to X_T and

$$|u|_{X_T} \leq C_1 |f|_{L^2(0,T;H)}. \tag{3.6}$$

(ii) For every $T > 0$ and every process $\xi \in M^2(0,T;V)$ a process $u = S \diamond \xi$ defined by

$$u(t) = \int_0^T S(t - r)\xi(r) \, dW(r), \;\; t \in [0,T]$$

belongs to $M^2(X_T)$ and

$$|u|_{M^2(X_T)} \leq C_2 |\xi|_{M^2(0,T;V)}. \tag{3.7}$$

(iii) For every $T > 0$ and every $u_0 \in V$, a function $u = Su_0$ defined by

$$u(t) = S(t)u_0, \;\; t \in [0,T]$$

belongs to X_T. Moreover, for every $T_0 > 0$ there exist $C_0 > 0$ such that for all $T \in (0, T_0]$,

$$|u|_{X_T} \leq C_0 \|u_0\|. \tag{3.8}$$

Now let us consider a semigroup $S(t)$, $t \in [0, \infty)$ as above and the abstract SPDEs

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F(s)ds + \int_0^t S(t-s)G(s)dW(s), \quad \text{for any } t > 0 \tag{3.9}$$

which is a mild version of the problem

$$\begin{cases} du(t) &= Au(t)\,dt + F\big(u(t)\big)\,dt + G\big(u(t)\big)dW(t), \quad t > 0, \\ u(0) &= u_0. \end{cases} \tag{3.10}$$

Definition 3.1. Assume that a V-valued $\mathcal{F}_0$ measurable random variable u_0 such that $\mathbb{E}\|u_0\|^2 < \infty$ is given. A local mild solution to problem (3.10) (with the initial time 0) is a pair (u, τ) such that

(1) τ is an accessible $\mathbb{F}$ stopping time,
(2) $u : [0, \tau) \times \Omega \to V$ is an admissible[2] process,
(3) there exists an approximating sequence $(\tau_m)_{m \in \mathbb{N}}$ of $\mathbb{F}$ finite stopping times such that $\tau_m \nearrow \tau$ a.s. and, for every $m \in \mathbb{N}$ and $t \geq 0$, we have

$$\mathbb{E}\Big(\sup_{s \in [0, t\wedge\tau_m]} \|u(s)\|^2 + \int_0^{t\wedge\tau_m} |u(s)|_E^2\, ds \Big) < \infty, \tag{3.11}$$

$$\begin{aligned} u(t\wedge\tau_m) &= S(t\wedge\tau_m)u_0 + \int_0^{t\wedge\tau_m} S(t\wedge\tau_m - s)F(u(s))ds \\ &+ \int_0^\infty \mathbb{1}_{[0, t\wedge\tau_m)} S(t\wedge\tau_m - s)G(u(s))\,dW(s). \end{aligned} \tag{3.12}$$

Along the lines of the paper [3], we said that a local solution $u(t)$, $t < \tau$ is called global iff $\tau = \infty$ a.s.

Remark 3.2. The Definition 3.1 of a local solution is independent of the choice of the sequence (σ_n). A proof of this fact follows from the continuity of trajectories of the process u (what is a consequence of admissibility of u) and is based on the following three principles.

(i) If τ is an accessible stopping time then there exist an increasing sequence τ_n of *discrete* stopping times such that a.s. $\tau_n < \tau$ and $\tau_n \nearrow \tau$;
(ii) if τ is an accessible stopping time and $\sigma \leq \tau$ is a stopping time then σ is also accessible.
(ii) if a pair (u, τ) is a local solution to (3.9), then (3.12) holds with t being any discrete stopping time.

It follows that the following is an equivalent definition of a local solution.
A pair (u, τ), where τ be an accessible stopping time and $u : [0, \tau) \times \Omega \to V$ is an admissible process, is a local mild solution to equation (3.10) iff for every accessible stopping time σ such that $\sigma < \tau$, for every $t \geq 0$, a.s.

$$\begin{aligned} u(t\wedge\sigma) &= S(t\wedge\sigma)u_0 + \int_0^{t\wedge\sigma} S(t\wedge\sigma - s)F(u(s))ds \\ &+ \int_0^\infty \mathbb{1}_{[0, t\wedge\sigma]} S(t\wedge\sigma - s)G(u(s))dW(s). \end{aligned} \tag{3.13}$$

Let us first formulate the following useful result.

Proposition 3.3. *Assume that a pair (u, τ) is a local mild solution to problem (3.10), where u_0 is an V-valued $\mathcal{F}_0$ measurable random variable such that $\mathbb{E}\|u_0\|^2$. Then for every finite stopping time σ, a pair $(u_{|[0,\tau\wedge\sigma)\times\Omega}, \tau\wedge\sigma)$ is also a local mild solution to problem (3.10).*

[2]This also follows from condition (3) below.

Let us recall following result, see [14, Lemmata III 6A and 6B].

Lemma 3.4. ***(The Amalgamation Lemma)*** *Let* A_1 *be a family of accessible stopping times with values in* $[0, \infty]$. *Then a function*

$$\tau := \sup\{\alpha : \alpha \in \mathrm{A}_1\}$$

is an accessible stopping time with values in $[0, \infty]$ *and there exists an* A_1*-valued increasing sequence* $\{\alpha_n\}_{n=1}^{\infty}$ *such that* τ *is the poitwise limit of* α_n.
Assume also that for each $\alpha \in \mathrm{A}_1$, $I_\alpha : [0, \alpha) \times \Omega \to V$ *is an admissible process such that for all* $\alpha, \beta \in \mathrm{A}_1$ *and every* $t > 0$,

$$I_\alpha(t) = I_\beta(t) \text{ a.s. on } \Omega_t(\alpha \wedge \beta). \tag{3.14}$$

Then, there exists an admissible process $\mathbf{I} : [0, \tau) \times \Omega \to V$, *such that every* $t > 0$,

$$\mathbf{I}(t) = I_\alpha(t) \text{ a.s. on } \Omega_t(\alpha). \tag{3.15}$$

Moreover, if $\tilde{I} : [0, \tau) \times \Omega \to X$ *is any process satisfying* (3.15) *then the process* $\tilde{I}$ *is a version of the process* I, *i.e. for any* $t \in [0, \infty)$

$$\mathbb{P}\left(\left\{\omega \in \Omega : t < \tau(\omega),\ I(t,\omega) \neq \tilde{I}(t,\omega)\right\}\right) = 0. \tag{3.16}$$

In particular, if in addition $\tilde{I}$ *is an admissible process, then*

$$\mathbf{I} = \tilde{I}. \tag{3.17}$$

Remark 3.5. Let us note that because both processes $\mathbf{I} : [0, \tau) \times \Omega \to V$ and $I_\alpha : [0, \alpha) \times \Omega \to V$ are admissible (and hence with almost sure continuous trajectories), and since $\alpha \le \tau$, condition (3.15) is equivalent to the following one:

$$\mathbf{I}_{|[0,\alpha)\times\Omega} = I_\alpha. \tag{3.18}$$

Similarly, condition (3.14) is equivalent to the following one

$$I_{\alpha|[0,\alpha\wedge\beta)\times\Omega} = I_{\beta|[0,\alpha\wedge\beta)\times\Omega}. \tag{3.19}$$

Definition 3.6. Consider a family $\mathcal{LS}$ of all local mild solution (u, τ) to the problem (3.10). For two elements $(u, \tau), (v, \sigma) \in \mathcal{LS}$ we write that $(u, \tau) \preceq (v, \sigma)$ iff $\tau \le \sigma$ a.s. and $v_{|[0,\tau)\times\Omega} \sim u$. Note that if $(u, \tau) \preceq (v, \sigma)$ and $(v, \sigma) \preceq (u, \tau)$, then $(u, \tau) \sim (v, \sigma)$. We write $(u, \tau) \prec (v, \sigma)$ iff $(u, \tau) \preceq (v, \sigma)$ and $(u, \tau) \not\sim (v, \sigma)$. Then the pair $(\mathcal{LS}, \preceq)$ is a partially ordered set in which, according to the Amalgamation Lemma 3.4, every non-empty chain has an upper bound.
Each such a maximal element (u, τ) in the set $(\mathcal{LS}, \preceq)$ is called a maximal local mild solution to the problem (3.10).
If (u, τ) is a maximal local mild solution to equation (3.10), the stopping time τ is called its lifetime.

A priori, there may be many maximal elements in $(\mathcal{LS}, \preceq)$ and hence many maximal local mild solutions to the problem (3.10). However, as we will see later, if the uniqueness of local solutions holds, the uniqueness of the maximal local mild solution will follow.

Remark 3.7. The following is an equivalent version of Definition 3.6. For a local mild solution (u, τ) the following conditions are equivalent.

(nm1) The pair (u, τ) is not a maximal local mild solution to problem (3.10).
(nm2) There exists a local mild solution (v, σ) to problem (3.10) such that $(u, \tau) \prec v, \sigma)$.
(nm3) There exists a local mild solution (v, σ) to problem (3.10) such that $\tau \le \sigma$ a.s., $v_{|[0,\tau)\times\Omega} \sim u$ and $\mathbb{P}(\tau < \sigma) > 0$.
(nm4) Every local mild solution (v, σ) to problem (3.10) such that $(u, \tau) \not\sim (v, \sigma)$ satisfies $(u, \tau) \not\prec (v, \sigma)$.

Definition 3.8. A local solution (u, τ) to problem (3.10) is unique iff for any other local solution (v, σ) to (3.10) the restricted processes $u_{[0,\tau\wedge\sigma)\times\Omega}$ and $v_{[0,\tau\wedge\sigma)\times\Omega}$ are equivalent.

Proposition 3.9. *Suppose that u_0 is an $\mathcal{F}_0$-measurable and p-integrable V-valued random variable. Suppose that there exist at least one local solution (u^0, τ^0) to problem (3.10) and that for any two local solutions (u^1, τ^2) and (u^1, τ^2), and every $t > 0$,*

$$u^1(t) = u^2(t) \text{ a.s. on } \{t < \tau^1 \wedge \tau^2\}. \tag{3.20}$$

Then there exists a maximal local mild solutions to the same problem.

Remark 3.10. Let us note that similarly to Remark 3.5, because both the local solutions u^1 and u^2 are admissible (and hence with almost sure continuous trajectories), condition (3.20) is equivalent to the following one:

$$u^1_{|[0,\tau^1\wedge\tau^2)\times\Omega} = u^2_{|[0,\tau^1\wedge\tau^2)\times\Omega}. \tag{3.21}$$

Proof of Proposition 3.9. Consider the family $\mathcal{LS}$ introduced in Definition 3.6. Note that in view of Proposition 3.3, if $(u, \tau) \in \mathcal{LS}$, then for any $T > 0$, $(u_{|[0,T\wedge\tau)\times\Omega}, T \wedge \tau)$ also belongs to $\mathcal{LS}$. Let $\mathcal{LC}$ be any nonempty chain in $\mathcal{LS}$ containing (u^0, τ^0). Set

$$\hat{\tau} := \sup\{\tau : (u, \tau) \in \mathcal{LC}\}.$$

Since the chain $\mathcal{LC}$ is non-empty, from the Amalgamation Lemma 3.4 it follows that $\hat{\tau}$ is an accessible stopping time and that there exists an admissible V-valued process $\hat{u}(t)$, $t < \hat{\tau}$ such that for all $(u, \tau) \in \mathcal{LC}$, $(\hat{u}_{|[0,\tau)\times\Omega}, \tau) \sim (u, \tau)$. In view of the Kuratowski-Zorn Lemma it remains to prove that the pair $(\hat{u}, \hat{\tau})$ belongs to $\mathcal{LS}$. To prove this let us consider an $\mathcal{LC}$-valued sequence, which exists in view of Lemma 3.4, (u^n, α_n) of local mild solutions to problem (3.10) such that $\alpha_n \nearrow \tau$ a.s. Moreover, by the above comment on Proposition 3.3 we can assume that α_n is a bounded from above stopping time. Moreover, since each (u^n, α_n) is a local mild solution, we can find a sequence (σ_n) of local times such that $\sigma_n < \alpha_n$, $\sigma_n \nearrow \tau$ a.s. and for each n, the condition (3.11) and equality (3.12) are satisfied by u^n.
Since $(u^n, \alpha_n) \in \mathcal{LC}$, we infer that $(\hat{u}_{|[0,\alpha_n)\times\Omega}, \alpha_n) \sim (u^n, \alpha_n)$ and thus we infer that also for each n, the condition (3.11) and equality (3.12) are satisfied by $\hat{u}$. This proves that $(\hat{u}, \hat{\tau}) \in \mathcal{LC}$. The proof is complete. □

Now we shall prove a counterparts of Corollary 2.29 and Lemma 2.31 from [3].

Proposition 3.11. *Assume that the Assumption 3.1-Assumption 3.3 hold. If a pair (u, τ) is a maximal local solution then*

$$\mathbb{P}\Big(\{\omega \in \Omega : \tau(\omega) < \infty, \exists \lim_{t\nearrow\tau(\omega)} u(t)(\omega) \in V\}\Big) = 0 \tag{3.22}$$

and

$$\lim_{t\nearrow\tau} |u|_{X_t} = \infty \quad \mathbb{P} - a.s. \text{ on } \{\tau < \infty\}. \tag{3.23}$$

Proof of Proposition 3.11 (by contradiction). **Part I.** Assume that there exists $\Omega_1 \subset \Omega$ such that $\mathbb{P}(\Omega_1) > 0$ and such that for any $\omega \in \Omega_1$ we have $\tau(\omega) < \infty$ and $\lim_{t\nearrow\tau(\omega)} u(t)(\omega) = \bar{v}(\omega) \in X$. Let us take an increasing sequence $\{\sigma_n\}_{n=1}^\infty$ of stopping times such that $\sigma_n < \tau$ a.s. and $\sigma_n \nearrow \tau$ a.s. Since the function $f_n : \Omega \ni (\omega) \mapsto u(\sigma_n(\omega))(\omega) \in V$, is $\mathcal{F}_{\sigma_n}$-measurable, see e.g. [21, Proposition I.2.18], and hence $\mathcal{F}_\tau$-measurable, the set

$$\Omega_2 := \{\omega \in \Omega : \tau(\omega) < \infty \text{ and } \lim_n f_n(\omega) := \bar{v}(\omega) \text{ exists}\}$$

belongs to the σ-algebra $\mathcal{F}_\tau$ and the function $\bar{v}$ is $\mathcal{F}_\tau$-measurable. Note that obviously $\Omega_1 \subset \Omega_2$ so that $\mathbb{P}(\Omega_2) > 0$.
Let $v(t)$, $\tau(\omega) \le t < \tau_2(\omega)$, be the solution to (3.10) with initial condition $v_{\tau(\omega)}(\omega) = \bar{v}(\omega)1_{\Omega_2}(\omega)$. Then, by [3, Corollary 2.9] the process $\tilde{u}(t)$ defined by

$$\tilde{u}(t)(\omega) = \begin{cases} u(t)(\omega), & \text{if } t < \tau(\omega), \\ v(t)(\omega), & \text{if } \omega \in \Omega_2 \text{ and } t \in [\tau(\omega), \tau_2(\omega)) \end{cases}$$

is a solution to (3.10). Obviously this contradicts the maximality of the solution $u(t)$.

Part II. Let us assume that there exists $\varepsilon > 0$ such that

$$\mathbb{P}\big(\{\tau < \infty\} \cap \{\limsup_{t\nearrow\tau} |u|_{X_t} < \infty\}\big) = 4\varepsilon.$$

Since the function $t \mapsto |u|_{X_t}$ is increasing, it means that

$$\mathbb{P}\big(\{\tau < \infty\} \cap \{\sup_{t\in[0,\tau)} |u|_{X_t} < \infty\}\big) = 4\varepsilon > 0.$$

Hence there exists $R > 0$ such that

$$\mathbb{P}\big(\{\tau < \infty\} \cap \{\sup_{t<\tau_\infty} |u(t)|_{X_t} < R\}\big) = 3\varepsilon > 0.$$

Define (our definition implies that $\sigma_R = \tau$, possibly ∞, when the set $\{\omega \in \Omega : |u|_{X_t} < R, t \in [0,\tau)\}$ is empty)

$$\sigma_R = \inf\{t \in [0,\tau] : |u|_{X_t} \geq R\}.$$

It is known that σ_R is a predictable stopping time and the set $\tilde{\Omega} = \{\sigma_R = \tau\} \cap \{\tau < \infty\}$ is $\mathcal{F}_{\sigma_R}$-measurable. Note also that $\tilde{\Omega} = \{\tau < \infty\} \cap \{\sup_{t<\tau_\infty} |u(t)|_{X_t} < R\}$ and so

$$\mathbb{P}(\tilde{\Omega}) \geq 3\varepsilon.$$

Define next two processes x and y by

$$x(t) = 1_{\{t<\sigma_R\}} G(u(t)), \quad y(t) = 1_{\{t<\sigma_R\}} F(u(t)) \;\; t \geq 0.$$

Note that $x(t) = 0$ or $|u|_{X_t} < R$.

Since $F(0) = 0$ by assumptions, from (3.3) we infer that

$$|F(x)|_H \leq C\|x\|^{p+1-\alpha}|x|_E^\alpha, \;\; x \in E.$$

Similarly, from (3.4)

$$\|G(x)\| \leq C\Big[1 + |x|_E^\beta \|x\|^{k+1-\beta}\Big], \;\; x \in E.$$

Let us now fix $T > 0$ and calculate.

$$\begin{aligned}
\int_0^T |y(t)|_H^2\, dt &= \mathbb{E}\int_0^T 1_{[0,\sigma_R)}(t)|F(u(t))|_H^2\, dt \leq C^2 \int_0^{T\wedge\sigma_R} \|u(t)\|^{2(p+1-\alpha)}|u(t)|_E^{2\alpha}\, dt \\
&\leq C^2\Big[\sup_{t\in[0,T\wedge\sigma_R)} \|u(t)\|^{2(p+1-\alpha)} \int_0^{T\wedge\sigma_R} |u(t)|_E^{2\alpha}\, dt\Big] \\
&\leq C^2\Big[\sup_{t\in[0,T\wedge\sigma_R)} \|u(t)\|^{2(p+1-\alpha)} \Big(\int_0^{T\wedge\sigma_R} |u(t)|_E^2\, dt\Big)^\alpha T^{1-\alpha}\Big] \\
&\leq C^2 T^{1-\alpha}\Big[(1-\alpha) \sup_{t\in[0,T\wedge\sigma_R)} \|u(t)\|^{2(1+\frac{p}{1-\alpha})} + \alpha \int_0^{T\wedge\sigma_R} |u(t)|_E^2\, dt\Big] \\
&\leq C^2 T^{1-\alpha}\Big[|u|_{X_{T\wedge\sigma_R}}^{2(1+\frac{p}{1-\alpha})}\Big] \leq C^2 T^{1-\alpha} R^{2(1+\frac{p}{1-\alpha})}
\end{aligned}$$

In particular,

$$\mathbb{E}\int_0^T |y(t)|_H^2\, dt \leq \mathbb{E} C^2 T^{1-\alpha} R^{2(1+\frac{p}{1-\alpha})} < \infty.$$

In a very similar manner we can show that

$$\mathbb{E}\int_0^T \|x(t)\|_V^2\, dt < \infty.$$

This allows us to define a process v by

$$v(t) = S(t)u_0 + \int_0^t S(t-s)y(s)ds + \int_0^\infty \mathbb{1}_{[0,t)}S(t-s)x(s)\,dW(s), \ \ t \geq 0. \tag{3.24}$$

The process v is defined globally and it is V-valued continuous. Moreover, since $y = F(u)$ and $x = G(u)$, on $[0, \sigma_R)$, we infer by [3, Remark 2.27] that $u = v$ on $[0, \sigma_R)$. In particular, there exist, a.s.

$$\lim_{t \nearrow \sigma_R} u(t) \text{ in } V.$$

In particular, the above limit exists on the set $\tilde{\Omega}$. Thus, a.s. on $\tilde{\Omega}$,

$$\lim_{t \nearrow T} u(t) \text{ in } V.$$

□

3.2. **An abstract result.** In this subsection we want to establish a general theorem of existence of mild solution to the following abstract SPDEs

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))ds + \int_0^t S(t-s)G(u(s))dW(s), \quad \text{for any } t > 0, \tag{3.25}$$

where $S(t)$, $t \in [0,\infty)$ is a semigroup, F and G are nonlinear map satisfying Assumption 3.3, Assumption 3.1, and Assumption 3.2, respectively.

Let $\theta : \mathbb{R}_+ \to [0,1]$ be a $\mathcal{C}_0^\infty$ non increasing function such that

$$\inf_{x \in \mathbb{R}_+} \theta'(x) \geq -1, \quad \theta(x) = 1 \text{ iff } x \in [0,1] \quad \text{and } \theta(x) = 0 \text{ iff } x \in [2,\infty). \tag{3.26}$$

and for $n \geq 1$ set $\theta_n(\cdot) = \theta(\frac{\cdot}{n})$. Note that if $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a non decreasing function, then for every $x, y \in \mathbb{R}$,

$$\theta_n(x)h(x) \leq h(2n), \quad |\theta_n(x) - \theta_n(y)| \leq \frac{1}{n}|x-y|. \tag{3.27}$$

Proposition 3.12. *Let F be a nonlinear mapping satisfying Assumption 3.1. Let us consider a map*

$$\Phi_T^n = \Phi_T : X_T \ni u \mapsto \theta_n(|u|_{X_\cdot})F(u) \in L^2(0,T;H).$$

Then Φ_T^n is globally Lipschitz and moreover, for any $u_1, u_2 \in X_T$,

$$|\Phi_T^n(u_1) - \Phi_T^n(u_2)|_{L^2(0,T;H)} \leq C(2n)^{p+1}\Big[2nC + 1\Big]T^{(1-\alpha)/2}|u_1 - u_2|_{X_T}. \tag{3.28}$$

The proof is based on a proof from the paper [7] which in turn was based on a proof from papers by De Bouard and Debussche [10, 11]. For simplicity, we will write Φ_T instead of Φ_T^n.

Proof. Note that $\Phi_T(0) = 0$. Assume that $u_1, u_2 \in X_T$. Denote, for $i = 1, 2$,

$$\tau_i = \inf\{t \in [0,T] : |u_i|_{X_t} \geq 2n\}.$$

Note that by definition, if the set on the RHS above is empty, then $\tau_i = T$.

Without loss of generality we can assume that $\tau_1 \le \tau_2$. We have the following chain of inequalities/equalities

$$
\begin{aligned}
|\Phi_T(u_1) - \Phi_T(u_2)|_{L^2(0,T;H)} &= \Big[\int_0^T |\theta_n(|u_1|_{X_t})F(u_1(t)) - \theta_n(|u_2|_{X_t})F(u_2(t))|_H^2\,dt\Big]^{1/2} \\
&\quad \text{because for } i=1,2,\ \ \theta_n(|u_i|_{X_t}) = 0 \text{ for } t \ge \tau_2 \\
&= \Big[\int_0^{\tau_2} |\theta_n(|u_1|_{X_t})F(u_1(t)) - \theta_n(|u_2|_{X_t})F(u_2(t))|_H^2\,dt\Big]^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
&= \Big[\int_0^{\tau_2} \big|\big[\theta_n(|u_1|_{X_t}) - \theta_n(|u_2|_{X_t})\big]F(u_2(t)) + \theta_n(|u_1|_{X_t})\big[F(u_1(t)) - F(u_2(t))\big]\big|_H^2\,dt\Big]^{1/2} \\
&\le \Big[\int_0^{\tau_2} \big|\big[\theta_n(|u_1|_{X_t}) - \theta_n(|u_2|_{X_t})\big]F(u_2(t))|_H^2\,dt\Big]^{1/2} \\
&\quad + \Big[\int_0^{\tau_2} |\theta_n(|u_1|_{X_t})\big[F(u_1(t)) - F(u_2(t))\big]|_H^2\,dt\Big]^{1/2} =: A + B
\end{aligned}
$$

Next, since θ_n is Lipschitz with Lipschitz constant $2n$ we have

$$
\begin{aligned}
A^2 &= \int_0^{\tau_2} |\big[\theta_n(|u_1|_{X_t}) - \theta_n(|u_2|_{X_t})\big]F(u_2(t))|^2\,dt \\
&\le 4n^2C^2 \int_0^{\tau_2} \big[||u_1|_{X_t} - |u_2|_{X_t}|\big]^2 |F(u_2(t))|_H^2\,dt \\
&\quad \text{by Minkowski inequality} \\
&\le 4n^2C^2 \int_0^{\tau_2} |u_1 - u_2|_{X_t}^2 |F(u_2(t))|_H^2\,dt \le 4n^2C^2 \int_0^{\tau_2} |u_1 - u_2|_{X_T}^2 |F(u_2(t))|_H^2\,dt \\
&\le 4n^2C^2 |u_1 - u_2|_{X_T}^2 \int_0^{\tau_2} |F(u_2(t))|_H^2\,dt
\end{aligned}
$$

Next, by assumptions

$$
\begin{aligned}
\int_0^{\tau_2} |F(u_2(t))|_H^2\,dt &\le C^2 \int_0^{\tau_2} \|u(t)\|^{2p+2-2\alpha} |u(t)|_E^{2\alpha}\,dt \\
&\le C^2 \sup_{t\in[0,\tau_2]} \|u(t)\|^{2p+2-2\alpha} \Big(\int_0^{\tau_2} |u(t)|_E^2\,dt\Big)^{\alpha} \tau_2^{1-\alpha} \\
&\le C^2 \tau_2^{1-\alpha} |u|_{X_{\tau_2}}^{2p+2} \le C^2 \tau_2^{1-\alpha} (2n)^{2p+2}
\end{aligned}
$$

Therefore,

$$
A \le C^2 \tau_2^{(1-\alpha)/2} (2n)^{p+2} |u_1 - u_2|_{X_T}.
$$

Also, because $\theta_n(|u_1|_{X_t}) = 0$ for $t \geq \tau_1$, and $\tau_1 \leq \tau_2$, we have

$$\begin{aligned}
B &= \Big[\int_0^{\tau_2} |\theta_n(|u_1|_{X_t})\big[F(u_1(t)) - F(u_2(t))\big]|_H^2\, dt\Big]^{1/2} \\
&= \Big[\int_0^{\tau_1} |\theta_n(|u_1|_{X_t})\big[F(u_1(t)) - F(u_2(t))\big]|_H^2\, dt\Big]^{1/2} \\
&\quad \text{because } \theta_n(|u_1|_{X_t}) \leq 1 \text{ for } t \in [0,\tau_1) \\
&\leq \Big[\int_0^{\tau_1} |F(u_1(t)) - F(u_2(t))|_H^2\, dt\Big]^{1/2} \\
&\leq C\Big[\int_0^{\tau_1} \|u_1(t) - u_2(t)\|^2 \|u_1(t)\|^{2p-2\alpha} |u_1(t)|_E^{2\alpha}\, dt\Big]^{1/2} \\
&+ C\Big[\int_0^{\tau_1} |u_1(t) - u_2(t)|_E^{2\alpha} \|u_1(t) - u_2(t)\|^{2-2\alpha} \|u_2(t)\|^{2p}\, dt\Big]^{1/2} \\
&\leq C \sup_{t\in[0,\tau_1]} \|u_1(t) - u_2(t)\| \|u_1(t)\|^{p-\alpha} \Big[\int_0^{\tau_1} |u_1(t)|_E^{2\alpha}\, dt\Big]^{1/2}
\end{aligned}$$

$$\begin{aligned}
&+ C \sup_{t\in[0,\tau_1]} \|u_1(t) - u_2(t)\|^{1-\alpha} \|u_2(t)\|^p \Big[\int_0^{\tau_1} |u_1(t) - u_2(t)|_E^{2\alpha}\, dt\Big]^{1/2} \\
&\leq C \sup_{t\in[0,T]} \|u_1(t) - u_2(t)\| \sup_{t\in[0,\tau_1]} \|u_1(t)\|^{p-\alpha} \Big[\int_0^{\tau_1} |u_1(t)|_E^2\, dt\Big]^{\alpha/2} \tau_1^{(1-\alpha)/2} \\
&+ C \sup_{t\in[0,T]} \|u_1(t) - u_2(t)\|^{1-\alpha} \sup_{t\in[0,\tau_1]} \|u_2(t)\|^p \Big[\int_0^{\tau_1} |u_1(t) - u_2(t)|_E^2\, dt\Big]^{\alpha/2} \tau_1^{(1-\alpha)/2} \\
&\leq C|u_1 - u_2|_{X_T} |u_1|_{X_{\tau_1}}^p \tau_1^{(1-\alpha)/2} + C|u_1 - u_2|_{X_T} |u_2|_{X_{\tau_1}}^p \tau_1^{(1-\alpha)/2} \\
&\quad \text{because} |u_1|_{X_{\tau_1}} \leq 2n \text{ and } |u_2|_{X_{\tau_1}} \leq |u_2|_{X_{\tau_2}} \leq 2n \\
&\leq C\tau_1^{(1-\alpha)/2} |u_1 - u_2|_{X_T} \Big[|u_1|_{X_{\tau_1}}^p + |u_2|_{X_{\tau_1}}^p\Big] \leq C(2n)^{p+1} \tau_1^{(1-\alpha)/2} |u_1 - u_2|_{X_T}
\end{aligned}$$

Summing up, we proved

$$\begin{aligned}
|\Phi_T(u_1) - \Phi_T(u_2)|_{L^2(0,T;H)} &\leq C^2 \tau_2^{(1-\alpha)/2} (2n)^{p+2} |u_1 - u_2|_{X_T} + C(2n)^{p+1} \tau_1^{(1-\alpha)/2} |u_1 - u_2|_{X_T} \\
&= C(2n)^{p+1}\big[2nC + 1\big] \tau_2^{(1-\alpha)/2} |u_1 - u_2|_{X_T}
\end{aligned}$$

The proof is complete. □

In what follows the function Φ_T^n from Proposition 3.12 will be denoted by $\Phi_{F,T}^n$. The following result is a special case of Proposition 3.12 with $H = V$.

Corollary 3.13. *Let G be a nonlinear mapping satisfying Assumption 3.2. Define a map Φ_G^n by*

$$\Phi_G = \Phi_G^n = \Phi_{G,T}^n : X_T \ni u \mapsto \theta_n(|u|_{X_\cdot}) G(u) \in L^2(0,T;V). \tag{3.29}$$

Then $\Phi_{G,T}^n$ is globally Lipschitz and moreover, for any $u_1, u_2 \in X_T$,

$$|\Phi_{G,T}^n(u_1) - \Phi_{G,T}^n(u_2)|_{L^2(0,T;V)} \leq C_G (2n)^{k+1} \big[2nC_G + 1\big] T^{(1-\beta)/2} |u_1 - u_2|_{X_T}. \tag{3.30}$$

Proposition 3.14. *Let Assumption 3.1-Assumption 3.3 hold. Consider a map*

$$\Psi_T^n = \Psi_T : M^2(X_T) \ni u \mapsto Su_0 + S * \Phi_{F,T}^n(u) + S \diamond \Phi_{G,T}^n(u) \in M^2(X_T) \tag{3.31}$$

Then Ψ_T is globally Lipschitz and moreover, for any $u_1, u_2 \in M^2(X_T)$,

$$|\Psi_T^n(u_1) - \Psi_T^n(u_2)|_{M^2(X_T)} \leq \hat{C}(n) T^{[1-(\alpha\vee\beta)]/2} |u_1 - u_2|_{M^2(X_T)}, \tag{3.32}$$

where

$$\hat{C}(n) = C_1 C_F (2n)^{p+1}\big[2nC_F + 1\big] + C_2 C_G (2n)^{k+1}\big[2nC_G + 1\big].$$

Proof. For simplicity of notation we will write Ψ_T and not Ψ_T^n. We will also write Ψ_F (resp. Ψ_G) instead of $\Psi_{F,T}$ (resp. $\Psi_{G,T}$).

Obviously in view of Assumption 3.3 the map Ψ_T is well defined. Let us fix $u_1, u_2 \in M^2(X_T)$. Then

$$\begin{aligned}
|\Psi_T(u_1) - \Psi_T(u_2)|_{M^2(X_T)} &\le |S * \Phi_F(u_1) - S * \Phi_F(u_2)|_{M^2(X_T)} \\
&+ |S \diamond \Phi_G(u_1) - S \diamond \Phi_G(u_2)|_{M^2(X_T)} \\
&\le C_1|\Phi_F(u_1) - \Phi_F(u_2)|_{M^2(0,T;H)} + C_2|\Phi_G(u_1) - \Phi_G(u_2)|_{M^2(X_T)} \\
&\le \Big[C_1 C_F (2n)^{p+1}\big[2nC_F + 1\big]T^{(1-\alpha)/2} \\
&+ C_2 C_G (2n)^{k+1}\big[2nC_G + 1\big]T^{(1-\beta)/2}\Big] \\
&\le \hat{C}(n) T^{[1-(\alpha\vee\beta)]/2},
\end{aligned}$$

where

$$\hat{C}(n) = C_1 C_F (2n)^{p+1}\big[2nC_F + 1\big] + C_2 C_G (2n)^{k+1}\big[2nC_G + 1\big].$$

The proof is complete. □

The first main result of this subsection is given in the following theorem.

Theorem 3.15. *Suppose that Assumption 3.1-Assumption 3.3 hold. Then for every $\mathcal{F}_0$-measurable V-valued square integrable random variable u_0 there exits a local process $u = \big(u(t), t \in [0, T_1)\big)$ which is the unique local mild solution to our problem. Moreover, given $R > 0$ and $\varepsilon > 0$ there exists $\tau(\varepsilon, R) > 0$, such that for every $\mathcal{F}_0$-measurable V-valued random variable u_0 satisfying $\mathbb{E}\|u_0\|^2 \le R^2$, one has*

$$\mathbb{P}\big(T_1 \ge \tau(\varepsilon, R)\big) \ge 1 - \varepsilon.$$

Proof. Owing to Proposition 3.14 we can argue as in [7, Proposition 5.1] to prove the first part of the theorem. For any $n \in \mathbb{N}$, $T > 0$ and $u_0 \in L^2(\Omega, \mathbb{P}; V)$ let Ψ^n_{T,u_0} be the mapping from $M^2(X_T)$ defined by

$$\Psi^n_{T,u_0}(u) = S_t u_0 + \int_0^t S_{t-r}[\theta_n(|u|_{X_r})F(u(r))]dr + \int_0^t S_{t-r}[\theta_n(|u|_{X_r})G(u(r))]dW^0(r).$$

It follows from Proposition 3.12, Corollary 3.13 and Assumption 3.3 that Ψ^n_{T,u_0} maps $M^2(X_T)$ into itself. From Proposition 3.14 we deduce that Ψ^n_{T,u_0} is globally Lipschitz, moreover it is a strict contraction for small T. Therefore we can find $\delta > 0$ such that Ψ^n_{T,u_0} is $\frac{1}{2}$-contraction. For $k \in \mathbb{N}$ let $(t_k)_{k\in\mathbb{N}}$ be a sequence of times defined by $t_k = k\delta$. By the $\frac{1}{2}$-contraction property of Ψ^n_{T,u_0} we can find $u^{[n,1]} \in M^2(X_\delta)$ such that $u^{[n,1]} = \Psi^n_{\delta,u_0}(u^{[n,1]})$. Since $u^{[n,1]} \in M^2(X_\delta)$ it follows that $u^{[n,1]}$ is $\mathcal{F}_t$-measurable and $u^{[n,1]}(t) \in L^2(\Omega, \mathbb{P}; V)$ for any $t \in [0, \delta]$. Thus replacing u_0 with $u^{[n,1]}(\delta)$ and using the same argument as above we can find $u^{[n,2]} \in M^2(X_\delta)$ such that $u^{[n,2]} = \Psi^n_{\delta,u^{[n,1]}(\delta)}(u^{[n,2]})$. By induction we can construct a sequence $u^{[n,k]} \subset M^2(X_\delta)$ such that $u^{[n,k]} = \Psi_{\delta,u^{[n,k-1]}}(u^{[n,k]})$. Now let u^n be the process defined by $u^n(t) = u^{[n,1]}(t)$, $t \in [0, \delta)$, and for $k = [\frac{T}{\delta}] + 1$ and $0 \le t < \delta$, let $u^n(t + k\delta) = u^{[n,k]}(t)$. By construction $u^n \in M^2(X_T)$ and $u^n = \Psi^n_{T,u_0}(u^n)$, consequently u^n is a global solution to the truncated equation

$$u(t) = S_t u_0 + \int_0^t S_{t-r}[\theta_n(|u|_{X_r})F(u(r))]dr + \int_0^t S_{t-r}[\theta_n(|u|_{X_r})G(u(r))]dW(r).$$

Arguing exactly as in [7, Theorem 4.9] we can show that it is unique. Now let $(\tau_n)_{n\in\mathbb{N}}$ be the sequence of stopping times defined by

$$\tau_n = \inf\{t \in [0, T] : |u^n|_{X_t} \ge n\}.$$

By definition $\theta_n(|u^n|_{X_r}) = 1$ for $r \in [0, t \wedge \tau_n)$, hence

$$\theta_n(|u^n|_{X_r})F(u^n(r)) = F(u^n(r)), r \in [0, t \wedge \tau_n).$$

From [6, Lemma A.1] we infer that

$$\int_0^{t\wedge\tau_n} S_{t\wedge\tau_n - r}[\theta_n(|u|_{X_r})G(u(r))]dW(r) = \int_0^{t\wedge\tau_n} S_{t\wedge\tau_n - r}G(u^n(r))dW(r).$$

From these remarks we easily deduce that u^n satisfies

$$u^n(t \wedge \tau^n) = S_{t\wedge\tau_n}u_0 + \int_0^{t\wedge\tau_n} S_{t\wedge\tau_n - r}F(u^n(r))dr + \int_0^{t\wedge\tau_n} S_{t\wedge\tau_n - r}G(u^n(r))dW(r).$$

Since τ_n is an accessible stopping time it follows that the process $(u^n(t), t < \tau_n)$ is a local solution to Eq. (3.10). This ends the proof of the first part of the theorem.

For the second part we will follow the lines of [7, Theorem 5.3]. For this we fix $\varepsilon > 0$ and choose N such that $N \geq 2\varepsilon^{-1/2}$.

Thanks to Proposition 3.12, Corollary 3.13 and Assumption 3.3 we can deduce that there exist some positive constants $\tilde{C}_i, i = 1, \dots, 4$ such that for any $u \in M^2(X_T)$ we have

$$\left(\mathbb{E}|S * \Phi_T^n(u)|_{X_T}^2\right)^{\frac{1}{2}} \leq \tilde{C}_1\tilde{C}_2(2n)^{p+2}(2n\tilde{C}_2 + 1)T^{(1-\alpha)/2},$$
$$|S \diamond \Phi_G^n(u)|_{M^2(X_T)} \leq \tilde{C}_3\tilde{C}_4(2n)^{k+2}[2n\tilde{C}_4 + 1]T^{(1-\beta)/2}.$$

Since $\alpha, \beta \in [0, 1)$ we infer from these inequalities that there exists a sequence $(K_n(T))_n$ of numerical functions with $\lim_{T\to 0}\sup_n K_n(T) = 0$ and

$$|S * \Phi_T^n(u) + S \diamond \Phi_G^n(u)|_{M^2(X_T)} \leq K_n(T),$$

for any $u \in M^2(X_T)$. Let us put $n = NR$ for some "large" N to be chosen later and choose $\delta_1(\varepsilon, R) > 0$ such that $K_n(\delta_1(\varepsilon, R)) \leq R$. Let Ψ_T^n be the mapping defined by (3.31). Since $\mathbb{E}\|u_0\|^2 \leq R^2$, we infer by the Assumption 3.3 (namely (3.6)) that

$$\begin{aligned}|\Psi_T^n(u)|_{M^2(X_T)} \leq& C_0R + K_n(T),\\ \leq& (C_0 + 1)R,\end{aligned}$$

for any $T \leq \delta_1(\varepsilon, R)$. That is, for $T \leq \delta_1(\varepsilon, R)$ the range of Ψ_T^n is included in the ball centered at 0 and of radius $(C_0 + 1)R$ of $M^2(X_T)$. Furthermore, Propositions 3.14 implies that there exists $C > 0$ such that for any $u_1, u_2 \in M^2(X_T)$

$$|\Psi_T^n(u_1) - \Psi_T^n(u_2)|_{M^2(X_T)} \leq CN^{p+1}R^{p+1}(NRC + 1)T^{]1-(\alpha\vee\beta)]/2}|u_1 - u_2|_{M^2(X_T)}.$$

Hence we can find $\delta_2(\varepsilon, R) > 0$ such that Ψ_T^n is a strict contraction for any $T \leq \delta_2(\varepsilon, R)$. Thus if one puts $\tau(\varepsilon, R) = \delta_1(\varepsilon, R) \wedge \delta_2(\varepsilon, R)$, the mapping Ψ_T^n has a unique fixed point u^n which satisfies

$$\mathbb{E}|u^n|_{X_{\tau(\varepsilon,R)}}^2 \leq (C_0 + 1)^2R^2.$$

As in [7, Proposition 5.1] we can show that $(u^n(t), t \leq \tau_n)$ is a local solution to problem (3.25). By the definition of the stopping time τ_n the set $\{\tau_n \leq \tau(\varepsilon, R)\}$ is contained in the set $\{|u^n|_{X_{\tau(\varepsilon,R)}} \geq n\}$. By Chebychev's inequality we have

$$\begin{aligned}\mathbb{P}(\tau_n \leq \tau(\varepsilon, R)) \leq& \mathbb{P}(|u^n|_{X_{\tau(\varepsilon,R)}}^2 \geq n),\\ \leq& (C_0 + 1)^2N^{-2},\\ \leq& \varepsilon,\end{aligned}$$

for $N \geq (C_0 + 1)\varepsilon^{\frac{1}{2}}$. Therefore for $N \geq (C_0 + 1)\varepsilon^{\frac{1}{2}}$ we have $\mathbb{P}(\tau_n \geq \tau(\varepsilon, R)) \geq 1 - \varepsilon$ and the stopping time $T_1 = \tau_n$ satisfies the requirements of the Theorem; this concludes the proof. $\square$

The next result is some kind of characterization of the maximal solution around its lifespan.

Theorem 3.16. *For every $u_0 \in L^2(\Omega, \mathcal{F}_0, V)$, the process $u = (u(t)\,,\, t < \tau_\infty)$ defined above is the unique local maximal solution to our equation. Moreover, $\mathbb{P}(\{\tau_\infty < \infty\} \cap \{\sup_{t<\tau_\infty} |u(t)|_V < \infty\}) = 0$ and on $\{\tau_\infty < \infty\}$, $\limsup_{t\to\tau_\infty} |u(t)|_V = +\infty$ a.s.*

Proof. Since $\tau_n \nearrow \tau_\infty$ we have $\tau_n \le \tau_\infty < \infty$ for any n on $\{\tau_\infty < \infty\}$. This implies in particular that for any $n > 0$ there exist $0 < t_n < \tau_\infty$, $n_0 > 0$ and $\delta > 0$ such that $\tau_\infty - t_n < \delta$ and $|u|_{X_{t_n}} \ge n$ for any $n \ge n_0$. This concludes that $u = (u(t)\,,\, t < \tau_\infty)$ is the unique maximal solution.

To prove the second statement we argue by contradiction. Assume that for some $\varepsilon > 0$, $\mathbb{P}(\{\tau_\infty < \infty\} \cap \{t \in [0, \tau_\infty) |u(t)|_V < \infty\}) = 4\varepsilon > 0$. Let $R > 0$ and assume that

$$\mathbb{P}(\{\tau_\infty < \infty\} \cap \{|u(t)|_V < R \text{ for all } t \in [0, \tau_\infty)\}) \ge 3\varepsilon.$$

Let $\sigma_R = \inf\{t \in [0, \tau_\infty) : |u(t)|_V \ge R\}$ and $\tilde{\Omega} = \{\sigma_R = \tau_\infty < \infty\}$.

Note that

$$\tilde{\Omega} = \{\tau_\infty < \infty\} \cap \{|u(t)|_V < R \text{ for all } t \in [0, \tau_\infty)\}$$

With this R and previous ε we can choose a number $\tau(\varepsilon, R) > 0$ as in Theorem 3.15. Let us now choose α such that $\alpha = \frac{1}{2}\tau(\varepsilon, R)$. By construction $\tau_n \nearrow \tau_\infty$ almost surely and hence for arbitrary $\delta > 0$ there exists $n_0 > 0$ such that $\mathbb{P}(\Omega_0) \ge (1 - \delta)\mathbb{P}(\tilde{\Omega})$, where $\Omega_0 := \{\omega \in \tilde{\Omega} : \tau_\infty - \tau_{n_0} < \alpha\}$. Choosing $\delta = \frac{1}{3}$ we get $\mathbb{P}(\Omega_0) \ge 2\varepsilon$.

Let $T_0 = \tau_{n_0}$ and

$$y_0 = \begin{cases} u(T_0) \text{ on } \Omega_0, \\ 0 \text{ otherwise.} \end{cases}$$

Note that $\mathbb{E}(|y_0|_V^2) \le R^2$. Then thanks to Theorem 3.15 the problem (3.25) with initial condition (starting at T_0) y_0 has a unique local solution denoted by $y(t)$, $t \in [T_0, T_0 + T_1)$. Moreover, the lifespan T_1 of the process $y(\cdot)$ satisfies $\mathbb{P}(T_1 \ge \tau(\varepsilon, R)) > 1 - \varepsilon$. From the first part of our theorem y is the unique maximal solution of (3.25) with initial condition y_0. Note that $\mathbb{P}(\tau_\infty - T_0 < \frac{1}{2}\tau(\varepsilon, R)) \ge 2\varepsilon$. Hence we infer that

$$\mathbb{P}(\Omega_1) \ge \varepsilon > 0,$$

where

$$\Omega_1 := \Omega_0 \cap \{T_1 \ge \tau(\varepsilon, R)\}.$$

Next we define a local stochastic process v by

$$v(t, \omega) = \begin{cases} u(t, \omega) \text{ if } \omega \in \Omega_1^c, \\ y(t, \omega) \text{ if } \omega \in \Omega_1 \text{ and } t > T_0, \\ u(t, \omega) \text{ if } \omega \in \Omega_1 \text{ and } t \in [0, T_0], \end{cases}$$

The process $v(\cdot)$ is a local solution of (3.25) with initial condition u_0. Since $\mathbb{P}(T_0 + \frac{1}{2}\tau(\varepsilon, R) > \tau_\infty) \ge \varepsilon$ the process $v(\cdot)$ satisfies

$$\begin{aligned}
\mathbb{E}\left(|v|_{X_{\tau_\infty + \frac{1}{2}\tau(\varepsilon,R)}} \cdot \mathbb{1}_{\Omega_1}\right) &\le \mathbb{E}\left(|v|_{X_{T_0+\tau(\varepsilon,R)}} \cdot \mathbb{1}_{\Omega_1}\right), \\
&\le \mathbb{E}\left[\left(|u|_{X_{T_0}} + |\mathbb{1}_{[T_0, T_0+\tau(\varepsilon,R)]} y|_{X_{T_0+\tau(\varepsilon,R)}}\right) \cdot \mathbb{1}_{\Omega_1}\right], \\
&\le \mathbb{E}\left(|u|_{X_{T_0}} \mathbb{1}_{\Omega_1}\right) + \mathbb{E}\left(\mathbb{1}_{\Omega_1} |\mathbb{1}_{[T_0, T_0+\tau(\varepsilon,R)]} y|_{X_{T_0+\tau(\varepsilon,R)}}\right), \\
&\le \mathbb{E}\left(|u|_{X_{T_0}} \mathbb{1}_{\Omega_0}\right) + \mathbb{E}\left(\mathbb{1}_{\Omega_1} |y|_{X_{[T_0, T_0+\tau(\varepsilon,R)]}}\right),
\end{aligned}$$

where the space $X_{[a,b]}$ and the norm $|\cdot|_{X_{[a,b]}}$ are defined similarly to the space X_T and norm $|\cdot|_{X_T}$.

Since $T_0 = \inf\{t \in [0, \tau_\infty) : |u|_{X_t} \geq n_0\} = \tau_{n_0}$, we infer that T_0 is finite ($T_0 < \tau_\infty$ to be precise) on the set Ω_0 and hence $|u|_{X_{T_0}} = n_0$ on the set Ω_0. Therefore, the first expected value $\mathbb{E}\left(|u|_{X_{T_0}} 1\!\!1_{\Omega_0}\right)$ is finite.

Since the solution $y(\cdot)$ is such that $1\!\!1_{[T_0, T_0+T_1)} y \in M^2_{loc}(X_{T_0+T_1})$ and the lifespan T_1 of $y(\cdot)$ satisfies $\mathbb{P}(T_1 + T_0 \geq T_0 + \tau(\varepsilon, R)) \geq \varepsilon$ we infer that

$$\mathbb{E}\left(|1\!\!1_{[T_0, T_0+\tau(\varepsilon,R)]} y|_{X_{T_0+\tau(\varepsilon,R)}}\right) \leq \mathbb{E}\left(|1\!\!1_{[T_0, T_0+\tau(\varepsilon,R)]} y|^2_{X_{T_0+\tau(\varepsilon,R)}}\right) < \infty.$$

Therefore

$$\mathbb{E}\left(|v|_{X_{\tau_\infty + \frac{1}{2}\tau(\varepsilon,R)}} \cdot 1\!\!1_{\Omega_1}\right) < \infty.$$

This implies in particular that $|v(t)|_{X_{\tau_\infty + \frac{1}{2}\tau(\varepsilon,R)}} < \infty$ on $\Omega_1 \subset \{\tau_\infty < \infty\}$ which contradicts Proposition 3.11. $\square$

The last result is very important since a'priori we only know that $|u|_{X_t} \to \infty$ as $t \nearrow \tau_\infty$ on the set $\{\tau_\infty < \infty\}$.

The proof of the existence of a global solution could then follow the proof in [7, Theorem 8.12].

Appendix A. Some preliminary estimates

In this section we recall or establish some crucial estimates needed for the proof of our mains results.

First, let $n \leq 4$ and put $a = \frac{n}{4}$. Then the following estimates, valid for all $\mathbf{u} \in \mathbb{W}^{1,4}$, are special cases of Gagliardo-Nirenberg's inequalities:

$$\|\mathbf{u}\|_{\mathbb{L}^4} \leq \|\mathbf{u}\|^{1-a} \|\nabla \mathbf{u}\|^a, \tag{A.1}$$

$$\|\mathbf{u}\|_{\mathbb{L}^\infty} \leq \|\mathbf{u}\|^{1-a}_{\mathbb{L}^4} \|\nabla \mathbf{u}\|^a_{\mathbb{L}^4}. \tag{A.2}$$

The inequality (A.1) can be written in the spirit of the continuous embedding

$$\mathbb{H}^1 \subset \mathbb{L}^4. \tag{A.3}$$

It follows from (A.2) and (A.3) that for $\mathbf{u} \in D(A_1)$

$$\|\mathbf{u}\|_{\mathbb{L}^\infty} \leq \|\mathbf{u}\|^{1-a}_1 \|\nabla \mathbf{u}\|^a_2. \tag{A.4}$$

Next we give some properties of the bilinear form B_1 and B_2 defined in Section 2.

Lemma A.1. *The bilinear map $B_1(\cdot,\cdot)$ maps continuously $\mathbb{V} \times \mathbb{H}^1$ into $\mathbb{V}^*$ and*

$$\langle B_1(\mathbf{u}, \mathbf{v}), \mathbf{w}\rangle = b(\mathbf{u}, \mathbf{v}, \mathbf{w}), \text{ for any } \mathbf{u} \in \mathbb{V}, \mathbf{v} \in \mathbb{H}^1, \mathbf{w} \in \mathbb{V}, \tag{A.5}$$

$$\langle B_1(\mathbf{u}, \mathbf{v}), \mathbf{w}\rangle = -b(\mathbf{u}, \mathbf{w}, \mathbf{v}) \text{ for any } \mathbf{u} \in \mathbb{V}, \mathbf{v} \in \mathbb{H}^1, \mathbf{w} \in \mathbb{V}, \tag{A.6}$$

$$\langle B_1(\mathbf{u}, \mathbf{v}), \mathbf{v}\rangle = 0 \text{ for any } \mathbf{u} \in \mathbb{V}, \mathbf{v} \in \mathbb{V}, \tag{A.7}$$

$$\|B_1(\mathbf{u}, \mathbf{v})\|_{\mathbb{V}^*} \leq C_0 \|\mathbf{u}\|^{1-\frac{n}{4}} \|\nabla \mathbf{u}\|^{\frac{n}{4}} \|\mathbf{v}\|^{1-\frac{n}{4}} \|\nabla \mathbf{v}\|^{\frac{n}{4}}, \text{ for all } \mathbf{u} \in \mathbb{V}, \mathbf{v} \in \mathbb{H}^1. \tag{A.8}$$

With an abuse of notation, we again denote by $B_2(\cdot,\cdot)$ the restriction of $B_2(\cdot,\cdot)$ to $\mathbb{V} \times \mathbb{H}^2$.

Lemma A.2. *We have*

$$\langle B_2(\mathbf{v}, \mathbf{d}), \mathbf{d}\rangle = 0, \text{ for any } \mathbf{v} \in \mathbb{V}, \mathbf{d} \in \mathbb{H}^2, \tag{A.9}$$

$$\langle B_2(\mathbf{v}, \mathbf{d}_2), \Delta \mathbf{d}_1\rangle = \langle M(\mathbf{d}_1, \mathbf{d}_2), \mathbf{v}\rangle, \text{ for any } \mathbf{v} \in \mathbb{V}, \mathbf{d}_1, \mathbf{d}_2 \in \mathbb{H}^2. \tag{A.10}$$

Proof. Eq. (A.9) easily follows by integration-by-parts and by taking into account that $\nabla \cdot \mathbf{v} = 0$ and $\mathbf{v}$ is zero on the boundary.

Note that $\langle B_2(\mathbf{v}, \mathbf{d}_2), \Delta \mathbf{d}_1\rangle = b(\mathbf{v}, \mathbf{d}_2, \Delta \mathbf{d}_1)$ is well-defined for any $\mathbf{v} \in \mathbb{V}, \mathbf{d}_1, \mathbf{d}_2 \in \mathbb{H}^2$. Thus, taking into account that $\mathbf{v}$ vanishes on the boundary we can perform an integration-by-parts and deduce that the identity (A.10) holds. $\square$

Next we give some properties of the semigroups $\{\mathbb{S}_1(t) : t \geq 0\}$ and $\{\mathbb{S}_2(t) : t \geq 0\}$ generated by $-\mathrm{A}_1$ on $\mathbb{H}$ and $-A$ on $\mathbb{X}_{\frac{1}{2}}$, respectively.

Lemma A.3. *Let $T \in (0,\infty)$, $g \in L^2(0,T;\mathbb{X}_0)$ and*

$$\mathbf{u}(t) = \int_0^T \mathbb{S}_2(t-s)g(s)ds, = \sum_{k\in\mathbb{N}} \int_0^T e^{-\lambda(t-s)} g_k(s)\, ds, \;\; t \in [0,T]. \tag{A.11}$$

Then

$$\mathbf{u} \in C(0,T;\mathbb{X}_{\frac{1}{2}}) \cap L^2(0,T;\mathbb{X}_1),$$

and

$$\|\mathbf{u}(t)\|_{C(0,T;\mathbb{X}_{\frac{1}{2}})} + \|\mathbf{u}(t)\|_{L^2(0,T;\mathbb{X}_1)} \leq (1 + \max(T, 1 + \frac{1}{\lambda_2}))\|g(t)\|_{L^2(0,T;\mathbb{X}_0)}.$$

Proof. This result is well-known. We refer the reader to [31]. □

Similarly we have the following result. See, for instance, [36]

Lemma A.4. *Let $T \in (0,\infty)$, $\tilde{g} \in L^2(0,T;\mathbb{H})$ and*

$$\mathbf{v}(t) = \int_0^T \mathbb{S}_1(t-s)\tilde{g}(s)ds.$$

We have

$$\|\mathbf{v}(t)\|_{C(0,T;D(\mathrm{A}_1^{\frac{1}{2}}))} + \|\mathbf{v}(t)\|_{L^2(0,T;D(\mathrm{A}_1))} \leq (1 + \frac{1}{\mu_1})\|\tilde{g}\|_{L^2(0,T;\mathbb{H})},$$

where μ_1 is the smallest eigenvalues of the Stokes operator A_1.

Let ζ_1 be an element of $M^2(0,T;\mathbb{H})$ (resp. $\zeta_2 \in M^2(0,T;\mathbb{X}_{\frac{1}{2}})$) and consider the stochastic convolutions

$$\mathbf{w}_2(t) = \int_0^t \mathbb{S}_2(t-s)\zeta_2(s)dW_2(s),$$

and

$$\mathbf{w}_1(t) = \int_0^t \mathbb{S}_1(t-s)\zeta_1(s)dW_1(s).$$

Lemma A.5. *There exists $C > 0$ such that*

$$\mathbb{E}\|\mathbf{w}_i(t)\|_{C(0,T;\tilde{V})} + \mathbb{E}\|\mathbf{w}_i(t)\|_{L^2(0,T;\tilde{E})} \leq C\mathbb{E}\|\zeta_i(t)\|_{L^2(0,T;\tilde{V})},$$

where $(\tilde{V},\tilde{E}) = (D(\mathrm{A}_1^{\frac{1}{2}}), D(A_1))$ if $i=1$, and $(\tilde{V},\tilde{E}) = (D(A_1), \mathbb{X}_1)$ if $i=2$.

Proof. This result is also well-known, see for example [31, 36]. □

Appendix B. Estimates for the local solution to SLC

The existence of global strong solution in two dimension of the SLC relies on the next two propositions.

Proposition B.1. *Let $(\mathbf{v},\mathbf{d})$ be the local strong solution from Theorem 2.9. For any $t > 0$ and $q \geq 2$ there exists a constant $C(h,q)$ such that*

$$\sup_{0\leq s\leq t_k} \mathbb{E}\Big(\|\mathbf{v}(s)\|^q + \|\mathbf{d}(s)\|^q + \|\nabla\mathbf{d}(s)\|^q\Big) \leq \mathbb{E}\Big(\|\mathbf{v}_0\|^q + \|\mathbf{d}_0\|^q + \|\nabla\mathbf{d}_0\|^q\Big)e^{C(h,q)t_k},$$

and

$$\mathbb{E}\Big(\int_0^{t_k}\Big(\|\mathbf{v}\|^{q-2}\|\mathrm{A}_1^{\frac{1}{2}}\mathbf{v}\|^2 + \|\mathbf{d}\|^{q-2}\|\nabla\mathbf{d}\|^2 + \frac{1}{2}\|\nabla\mathbf{d}\|^{q-2}\|\Delta\mathbf{d}\|^2\Big)ds\Big)$$
$$\leq \mathbb{E}\Big(\|\mathbf{v}_0\|^q + \|\mathbf{d}_0\|^q + \|\nabla\mathbf{d}_0\|^q\Big)e^{C(h,q)t_k}.$$

Proof. Firstly, from the definitions of G^2, G and the Assumption 2.1 it is not difficult to check that

$$\|\mathbf{d}\|^{q-2}\Big(\frac{1}{2}\|G(\mathbf{d})\|^2+\langle \mathbf{d},G^2(\mathbf{d})\rangle\Big)\leq C(h)\|\mathbf{d}\|^q,$$
$$\|\mathbf{d}\|^{q-4}|\langle \mathbf{d},G(\mathbf{d})\rangle|^2\leq C(h)\|\mathbf{d}\|^q.$$

By straightforward calculation we see that there exists a constant $C(h)>0$ such that

$$\|\nabla_x G(\mathbf{d})\|^2 \leq C(h)(\|\nabla\mathbf{d}\|^2+\|\mathbf{d}\|^2), \tag{B.1}$$
$$\|\nabla_x G^2(\mathbf{d})\|^2 \leq C(h)(\|\nabla\mathbf{d}\|^2+\|\mathbf{d}\|^2), \tag{B.2}$$
$$|\langle\nabla_x G^2(\mathbf{d}),\nabla\mathbf{d}\rangle| \leq C(h)(\|\nabla\mathbf{d}\|^2+\|\mathbf{d}\|^2).$$

Using these estimates and Young's inequality we have that

$$\begin{aligned}\|\nabla\mathbf{d}\|^{q-2}\Big(\|\nabla_x G(\mathbf{d})\|^2+\langle\nabla_x G^2(\mathbf{d}),\nabla\mathbf{d}\rangle\Big) &\leq C(h)\|\nabla\mathbf{d}\|^{q-2}\left(\|\nabla\mathbf{d}\|^2+\|\mathbf{d}\|^2\right),\\ &\leq C(h,q)\|\nabla\mathbf{d}\|^q+\|\mathbf{d}\|^q.\end{aligned}$$

Similarly

$$\|\nabla\mathbf{d}\|^{q-4}|\langle\nabla_x G(\mathbf{d}),\nabla\mathbf{d}\rangle|^2\leq C(h,q)\|\nabla\mathbf{d}\|^q+\|\mathbf{d}\|^q.$$

From Young's inequality we deduce that

$$\begin{aligned}\|\nabla\mathbf{d}\|^{q-2}|\langle f(\mathbf{d}),\Delta\mathbf{d}\rangle| &\leq C\|\nabla\mathbf{d}\|^{q-2}\|\mathbf{d}\|^2+\frac{1}{2}\|\nabla\mathbf{d}\|^{q-2}\|\Delta\mathbf{d}\|^2,\\ &\leq C\left(\|\mathbf{d}\|^q+\|\nabla\mathbf{d}\|^q\right)+\frac{1}{2}\|\nabla\mathbf{d}\|^{q-2}\|\Delta\mathbf{d}\|^2,\end{aligned}$$

and

$$\|\mathbf{d}\|^{q-2}|-\langle f(\mathbf{d}),\mathbf{d}\rangle|\leq C\|\mathbf{d}\|^q.$$

From the assumption on S we easily derive that

$$\|\mathbf{v}\|^{q-2}\|S(\mathbf{v})\|^2_{\mathcal{J}_2(K_1,\mathrm{V})}\leq C(1+\|\mathbf{v}\|^q).$$

Secondly, let $t>0$ and $s\in[0,t_k]$. Using Itô's formula we have

$$\begin{aligned}d\|\mathbf{d}(s)\|^q = q\|\mathbf{d}(s)\|^{q-2}\Big[-\|\nabla\mathbf{d}(s)\|^2-\langle f(\mathbf{d}(s)),\mathbf{d}(s)\rangle+\frac{1}{2}\langle G^2(\mathbf{d}(s)),\mathbf{d}(s)\rangle+\frac{1}{2}\|G(\mathbf{d})\|^2\Big]ds\\ +q(q-2)\|\mathbf{d}(s)\|^{q-4}|\langle\mathbf{d}(s),G(\mathbf{d}(s))\rangle|^2ds.\end{aligned}$$

The function $\|\nabla\mathbf{d}(s)\|^q$ satisfies the Itô's formula

$$\begin{aligned}d\|\nabla\mathbf{d}(s)\|^q = q\|\nabla\mathbf{d}(s)\|^{q-2}\Big(-\|\Delta\mathbf{d}(s)\|^2+\langle B_2(\mathbf{v}(s)\cdot\nabla\mathbf{d}(s)-f(\mathbf{d}(s)),\Delta\mathbf{d}(s)\rangle\Big)ds\\ +q\|\nabla\mathbf{d}(s)\|^{q-2}\Big(\|\nabla_x G(\mathbf{d}(s))\|^2+\langle\nabla_x G^2(\mathbf{d}(s)),\nabla\mathbf{d}(s)\rangle\Big)ds\\ +q(q-2)\|\nabla\mathbf{d}(s)\|^{q-4}|\langle\nabla_x G(\mathbf{d}(s)),\nabla\mathbf{d}(s)\rangle|^2ds\\ +q\|\nabla\mathbf{d}(s)\|^{q-2}\langle\nabla_x G(\mathbf{d}(s)),\nabla\mathbf{d}(s)\rangle dW_2(s).\end{aligned}$$

From Itô's formula we obtain

$$\begin{aligned}d\|\mathbf{v}(s)\|^q\leq q\|\mathbf{v}(s)\|^{q-2}\Big(-\|\mathrm{A}_1^{\frac{1}{2}}\mathbf{v}(s)\|^2-\langle M(\mathbf{d}(s)),\mathbf{v}(s)\rangle+\|S(\mathbf{v}(s))\|^2_{\mathcal{J}_2(\mathrm{K}_1,\mathrm{V})}\Big)ds\\ +q(q-2)\|\mathbf{v}(s)\|^{q-2}\|S(\mathbf{v}(s))\|^2_{\mathcal{J}_2(\mathrm{K}_1,\mathrm{V})}ds+q\|\mathbf{v}(s)\|^{q-2}\langle\mathbf{v}(s),S(\mathbf{v}(s))\rangle dW_1(s).\end{aligned}$$

Thirdly, we want to estimate $\|\mathbf{d}(s)\|^q + \|\nabla\mathbf{d}(s)\|^q + \|\mathbf{v}(s)\|^q$. To do so we first add up side by side the Itô formula above and use (A.10) and the estimates we obtained in the first step of the proof. Integrating the result from this procedure implies that for any $t > 0$ and $s \in [0, t_k]$

$$\begin{aligned}\varphi(s_k) + q\int_0^{s_k}\Big(\|\mathbf{v}\|^{q-2}\|\mathrm{A}_1^{\frac{1}{2}}\mathbf{v}\|^2 + \|\mathbf{d}\|^{q-2}\|\nabla\mathbf{d}\|^2 + \frac{1}{2}\|\nabla\mathbf{d}\|^{q-2}\|\Delta\mathbf{d}\|^2\Big)dr \\ \le q\int_0^{s_k}\|\mathbf{v}\|^{q-2}\langle\mathbf{v}, S(\mathbf{v})\rangle dW_1(r) + q\int_0^{s_k}\|\nabla\mathbf{d}\|^{q-2}\langle\nabla_x G(\mathbf{d}), \nabla\mathbf{d}\rangle dW_2(r) \\ +\varphi(0) + C(h,q)\int_0^{s_k}\Big(\|\mathbf{v}\|^q + \|\mathbf{d}\|^q + \|\nabla\mathbf{d}\|^q\Big)dr,\end{aligned}$$

where

$$\begin{aligned}\varphi(t) :=& \|\mathbf{v}(t)\|^q + \|\mathbf{d}(t)\|^q + \|\nabla\mathbf{d}(t)\|^q, \\ \varphi(0) :=& \|\mathbf{v}_0\|^q + \|\mathbf{d}_0\|^q + \|\nabla\mathbf{d}_0\|^q\end{aligned}$$

Taking the mathematical expectation, the supremum over $s \in [0, t_k]$ in this last estimate and using Gronwall's inequality yields the sought estimates in the proposition. □

Proposition B.2. *let $n = 2$ and $(\mathbf{v}, \mathbf{d})$ be the local strong solution from Theorem 2.9 and*

$$\Psi(\mathbf{d}) = \|-A\mathbf{d} - f(\mathbf{d})\|^2.$$

There exist some positive constants $C(\mathbf{v}_0, \mathbf{d}_0)$ and $C(h) > 0$ such that for any $t > 0$

$$\begin{aligned}\mathbb{E}\Big[e^{-\int_0^{t_k}\phi(r)dr}\Big(\|\mathrm{A}_1^{\frac{1}{2}}\mathbf{v}(t_k)\|^2 + \Psi(\mathbf{d}(t_k))\Big)\Big] + \mathbb{E}\int_0^{t_k}\Phi(s)\Big(2\|\mathrm{A}_1\mathbf{v}\|^2 + \|\nabla(\Delta\mathbf{d} - f(\mathbf{d}))\|^2\Big)ds \\ \le \|\mathrm{A}_1^{\frac{1}{2}}\mathbf{v}_0\|^2 + \Psi(\mathbf{d}_0) + e^{C(h)t_k}t_k\Big(\|\mathrm{A}_1^{\frac{1}{2}}\mathbf{v}_0\|^2 + \Psi(\mathbf{d}_0) + C(\mathbf{v}_0, \mathbf{d}_0)t_k\Big),\end{aligned} \tag{B.3}$$

where $\phi(\cdot)$ is a positive function that will be in the course of the proof and $\Phi(s) = e^{-\int_0^s \phi(r)dr}$.

Proof. Since the local strong solution $(\mathbf{v}, \mathbf{d}) \in D(\mathrm{A}_1) \times \mathbb{X}_1$ almost surely we can view $\Psi(\mathbf{d})$ as

$$\Psi(\mathbf{d}) = \|\Delta\mathbf{d} - f(\mathbf{d})\|^2.$$

The functional $\Psi(\mathbf{d})$ is twice differentiable with first and second Fréchet derivatives defined by

$$\Psi'(\mathbf{d})(h) = 2\langle -A\mathbf{d} - f(\mathbf{d}), -Ah - f'(\mathbf{d})h\rangle,$$

and

$$\Psi''(\mathbf{d})[h,k] = 2\langle -Ak - f'(\mathbf{d})k, -Ah - f'(\mathbf{d})h\rangle - 2\langle -A\mathbf{d} - f(\mathbf{d}), f''(\mathbf{d})h\ k\rangle,$$

for any $h,\ k \in \mathbb{X}_{\frac{1}{2}}$.

Let us recall that

$$\begin{aligned}d\mathbf{v}(t) + \Big(\mathrm{A}_1\mathbf{v}(t) + B_1(\mathbf{v}(t), \mathbf{v}(t)) + M(\mathbf{d}(t))\Big)dt &= S(\mathbf{v}(t))dW_1, \\ d\mathbf{d}(t) + \Big(A\mathbf{d}(t) + B_2(\mathbf{v}(t), \mathbf{d}(t)) + f(\mathbf{d}(t)) - \frac{1}{2}G^2(\mathbf{d}(t))\Big)dt &= G(\mathbf{d}(t))dW_2.\end{aligned}$$

Since $(\mathbf{v}, \mathbf{d}) \in X_t$ for any $t \in [0, \tau_\infty)$ we see that

$$\mathrm{A}_1\mathbf{v}(t) + B_1(\mathbf{v}(t), \mathbf{v}(t)) + M(\mathbf{d}(t)) \in L^2(0, t; \mathbb{H}),$$

and

$$A\mathbf{d}(t) + B_2(\mathbf{v}(t), \mathbf{d}(t)) + f(\mathbf{d}(t)) - \frac{1}{2}G^2(\mathbf{d}(t)) \in L^2(0, t; \mathbb{X}_0),$$

for any $t \in [0, \tau_\infty)$. Therefore by considering the Gelfand triples $D(\mathrm{A}_1) \subset D(\mathrm{A}_1^{\frac{1}{2}}) \subset \mathbb{H}$ and $\mathbb{X}_1 \subset \mathbb{X}_{\frac{1}{2}} \subset \mathbb{X}_0$ Itô's formula for the functional $\|\mathrm{A}_1^{\frac{1}{2}}\mathbf{v}(t \wedge \tau_k)\|^2$ and $\Psi(\mathbf{d}(t \wedge \tau_k))$ for any integer

$k > 0$ are applicable to our situation (see [31, Theorem I.3.3.2, Page 147]). By Itô's formula we have

$$\|\nabla \mathbf{v}(t_k)\|^2 - \|\nabla \mathbf{v}_0\|^2 = -2\int_0^{t_k} \langle \mathrm{A}_1\mathbf{v} + B_1(\mathbf{v},\mathbf{v}) + \Pi\nabla\cdot(\nabla\mathbf{d}\otimes\nabla\mathbf{d}), \mathrm{A}_1\mathbf{v}\rangle ds$$
$$+\int_0^{t_k} \|S(\mathbf{v})\|^2_{\mathcal{J}_2(\mathrm{K}_1,\mathbb{V})} ds + 2\int_0^{t_k} \langle \mathrm{A}_1^{\frac12} S(\mathbf{v}), \mathrm{A}_1^{\frac12}\mathbf{v}\rangle dW_1(s).$$

As in Lin and Liu 1995 [24] we use the identity

$$\begin{aligned}\langle \Pi\nabla\cdot(\nabla\mathbf{d}\otimes\nabla d), \mathrm{A}_1\mathbf{v}\rangle &= \langle \Delta\mathbf{d}\nabla\mathbf{d}, \mathrm{A}_1\mathbf{v}\rangle + \langle \nabla\left(\frac{|\nabla\mathbf{d}|^2}{2}\right), \mathrm{A}_1\mathbf{v}\rangle,\\ &= \langle (\Delta\mathbf{d} - f(\mathbf{d}))\nabla\mathbf{d}, \mathrm{A}_1\mathbf{v}\rangle + \langle f(\mathbf{d})\nabla\mathbf{d}, \mathrm{A}_1\mathbf{v}\rangle.\end{aligned}$$

To get the second line of this identity we have used the fact that $\Delta\mathbf{v}$ is divergence free. As a consequence of this we have that

$$\begin{aligned}\|\mathrm{A}_1^{\frac12}\mathbf{v}(t_k)\|^2 - \|\mathrm{A}_1^{\frac12}\mathbf{v}_0\|^2 = &-2\int_0^{t_k} \langle \mathrm{A}_1\mathbf{v} + B_1(\mathbf{v},\mathbf{v}) + f(\mathbf{d})\nabla\mathbf{d}, \mathrm{A}_1\mathbf{v}\rangle ds + \int_0^{t_k} \|S(\mathbf{v})\|^2_{\mathcal{J}_2(\mathrm{K}_1,\mathbb{V})} ds\\ &-2\int_0^{t_k} \langle (\Delta\mathbf{d} - f(\mathbf{d}))\nabla\mathbf{d}, \mathrm{A}_1\mathbf{v}\rangle ds + 2\int_0^{t_k} \langle \mathrm{A}_1^{\frac12} S(\mathbf{v}), \mathrm{A}_1^{\frac12}\mathbf{v}\rangle dW_1(s).\end{aligned} \tag{B.4}$$

At the same time we have

$$\begin{aligned}\Psi(\mathbf{d}(t_k)) - \Psi(\mathbf{d}_0) = &\int_0^{t_k} \Psi'(\mathbf{d})[-\mathbf{v}\cdot\nabla\mathbf{d} - A\mathbf{d} - f(\mathbf{d})] ds + \frac12\int_0^{t_k} \Psi'(\mathbf{d})[G^2(\mathbf{d})] ds\\ &+\frac12\int_0^{t_k} \Psi''(\mathbf{d})[G(\mathbf{d})G(\mathbf{d})] ds + \int_0^{t_k} \Psi'(\mathbf{d})[G(\mathbf{d})] dW_2(s).\end{aligned}$$

Using the definition of $\Psi'(\mathbf{d})$ we see from this last equation that

$$\begin{aligned}\Psi(\mathbf{d}(t_k)) - \Psi(\mathbf{d}_0) = &\,2\int_0^{t_k} \langle -A\mathbf{d} - f(\mathbf{d}), -A\Big(-\mathbf{v}\cdot\nabla\mathbf{d} - A\mathbf{d} - f(\mathbf{d})\Big)\rangle ds\\ &+\frac12\int_0^{t_k} \Psi'(\mathbf{d})[G^2(\mathbf{d})] ds + \frac12\int_0^{t_k} \Psi''(\mathbf{d})[G(\mathbf{d}), G(\mathbf{d})] ds\\ -2\int_0^{t_k} \langle -A\mathbf{d} - f(\mathbf{d}), f'(\mathbf{d})\Big(-\mathbf{v}\cdot\nabla\mathbf{d} - A\mathbf{d} - f(\mathbf{d})\Big)\rangle ds &+ \int_0^{t_k} \Psi'(\mathbf{d})[G(\mathbf{d})] dW_2(s)\\ = &\,2\sum_{i=1}^4 \int_0^{t_k} I_i(s) ds + \int_0^{t_k} \Psi'(\mathbf{d})[G(\mathbf{d})] dW_2(s).\end{aligned} \tag{B.5}$$

For $(\mathbf{v},\mathbf{d}) \in D(\mathrm{A}_1)\times\mathbb{X}_1$ almost surely we easily check that $-\mathbf{v}.\nabla\mathbf{d} \in D(A)$. Therefore I_1 can be rewritten in the following form

$$I_1(s) = \langle -\Delta\mathbf{v}\nabla\mathbf{d} - \mathbf{v}\cdot\nabla\Delta\mathbf{d}, \Delta\mathbf{d} - f(\mathbf{d})\rangle + \langle \Delta\mathbf{d} - f(\mathbf{d}), -A(\Delta\mathbf{d} - f(\mathbf{d}))\rangle,$$

where the first term is a product of two function in $\mathbb{L}^2$ and the second term is understood as the duality pairing between an element of $\mathbb{H}^1$ and $(\mathbb{H}^1)^*$. Invoking Eq. (2.1) we derive that

$$\begin{aligned}I_1(s) = &-\langle \mathbf{v}\cdot\nabla[f(\mathbf{d})], \Delta\mathbf{d} - f(\mathbf{d})\rangle - \|\nabla(\Delta\mathbf{d} - f(\mathbf{d}))\|^2\\ &+\langle -\Delta\mathbf{v}\nabla\mathbf{d} - \mathbf{v}\cdot\nabla(\Delta\mathbf{d} - f(\mathbf{d})), \Delta\mathbf{d} - f(\mathbf{d})\rangle.\end{aligned} \tag{B.6}$$

We derive from Eq. (B.4), Eq. (B.5) and Eq. (B.6) that

$$
\begin{aligned}
&\|\mathrm{A}_1^{\frac{1}{2}}\mathbf{v}(t_k)\|^2 + \Psi(\mathbf{d}(t_k)) - \|\mathrm{A}_1^{\frac{1}{2}}\mathbf{v}_0\|^2 - \Psi(\mathbf{d}_0) + 2\int_0^{t_k}\Big[\|\mathrm{A}_1\mathbf{v}\|^2 + \|\nabla(\Delta\mathbf{d} - f(\mathbf{d}))\|^2\Big]ds \\
&\quad = 2\int_0^{t_k}\langle B_1(\mathbf{v},\mathbf{v}) + \nabla\mathbf{d}f(\mathbf{d}), \Delta\mathbf{v}\rangle ds + 2\int_0^{t_k}\langle\nabla\mathbf{d}[\Delta\mathbf{d} - f(\mathbf{d})], \Delta\mathbf{v}\rangle ds \\
&\quad -2\int_0^{t_k}\langle\Delta\mathbf{v}\nabla\mathbf{d}, \Delta\mathbf{d} - f(\mathbf{d})\rangle ds - 2\int_0^{t_k}\langle\mathbf{v}\cdot\nabla[f(\mathbf{d})], \Delta\mathbf{d} - f(\mathbf{d})\rangle ds \\
&\quad -2\int_0^{t_k}\langle f'(\mathbf{d})\Big(-\mathbf{v}\cdot\nabla\mathbf{d} + \Delta\mathbf{d} - f(\mathbf{d})\Big), \Delta\mathbf{d} - f(\mathbf{d})\rangle ds \\
&\quad -2\int_0^{t_k}\langle\mathbf{v}\cdot\nabla[\Delta\mathbf{d} - f(\mathbf{d})], \Delta\mathbf{d} - f(\mathbf{d})\rangle ds + OT.
\end{aligned} \tag{B.7}
$$

where the other term OT is defined by

$$
\begin{aligned}
OT = &\int_0^{t_k}\|S(\mathbf{v})\|^2_{\mathcal{J}_2(\mathrm{K}_1,\mathrm{V})}ds + \frac{1}{2}\int_0^{t_k}\Psi'(\mathbf{d})[G^2(\mathbf{d})]ds + \frac{1}{2}\int_0^{t_k}\Psi''(\mathbf{d})[G(\mathbf{d}), G(\mathbf{d})]ds \\
&+2\int_0^{t_k}\langle\mathrm{A}_1^{\frac{1}{2}}S(\mathbf{v}), \mathrm{A}_1^{\frac{1}{2}}\mathbf{v}\rangle dW_1(s) + \int_0^{t_k}\Psi'(\mathbf{d})[G(\mathbf{d})]dW_2(s).
\end{aligned}
$$

Thanks to (A.7) the identity (B.7) can be simplified as follows

$$
\begin{aligned}
&\|\mathrm{A}_1^{\frac{1}{2}}\mathbf{v}(t_k)\|^2 + \Psi(\mathbf{d}(t_k)) - \|\mathrm{A}_1^{\frac{1}{2}}\mathbf{v}_0\|^2 - \Psi(\mathbf{d}_0) + 2\int_0^{t_k}\Big[\|\mathrm{A}_1\mathbf{v}\|^2 + \|\nabla(\Delta\mathbf{d} - f(\mathbf{d}))\|^2\Big]ds \\
&= 2\int_0^{t_k}\langle B_1(\mathbf{v},\mathbf{v}) + \nabla\mathbf{d}f(\mathbf{d}), \Delta\mathbf{v}\rangle ds - 2\int_0^{t_k}\langle f'(\mathbf{d})\Big(-\mathbf{v}\cdot\nabla\mathbf{d} + \Delta\mathbf{d} - f(\mathbf{d})\Big), \Delta\mathbf{d} - f(\mathbf{d})\rangle ds \\
&\quad -2\int_0^{t_k}\langle\mathbf{v}\cdot\nabla[f(\mathbf{d})], \Delta\mathbf{d} - f(\mathbf{d})\rangle ds + OT.
\end{aligned}
$$

From the last equality and the chain rule $\mathbf{v}\cdot\nabla[f(\mathbf{d})] = f'(\mathbf{d})\mathbf{v}\cdot\nabla\mathbf{d}$ we deduce that

$$
\begin{aligned}
&\|\mathrm{A}_1^{\frac{1}{2}}\mathbf{v}(t_k)\|^2 + \Psi(\mathbf{d}(t_k)) - \|\mathrm{A}_1^{\frac{1}{2}}\mathbf{v}_0\|^2 - \Psi(\mathbf{d}_0) + 2\int_0^{t_k}\Big[\|\mathrm{A}_1\mathbf{v}\|^2 + \|\nabla(\Delta\mathbf{d} - f(\mathbf{d}))\|^2\Big]ds \\
&= 2\int_0^{t_k}\langle B_1(\mathbf{v},\mathbf{v}) + \nabla\mathbf{d}f(\mathbf{d}), \Delta\mathbf{v}\rangle ds - 2\int_0^{t_k}\langle f'(\mathbf{d})\Big(\Delta\mathbf{d} - f(\mathbf{d})\Big), \Delta\mathbf{d} - f(\mathbf{d})\rangle ds + OT.
\end{aligned}
$$

To get rid of some terms we want to apply the Itô's formula to the function

$$
\begin{aligned}
\Xi(t_k, \mathbf{v}, \mathbf{d}) :=& \Phi(t_k)\left(\|\mathrm{A}_1^{\frac{1}{2}}\mathbf{v}(t_k)\|^2 + \Psi(\mathbf{d}(t_k))\right) \\
=& e^{-\int_0^{t_k}\phi(s)ds}\left(\|\mathrm{A}_1^{\frac{1}{2}}\mathbf{v}(t_k)\|^2 + \Psi(\mathbf{d}(t_k))\right),
\end{aligned} \tag{B.8}
$$

where $\phi(s)$ is going to be determined by some estimates on

$$\langle B_1(\mathbf{v},\mathbf{v}), \Delta\mathbf{v}\rangle + OT. \tag{B.9}$$

From this point let us study the terms in Eq. (B.9). First,

$$|\langle B_1(\mathbf{v},\mathbf{v}), \Delta\mathbf{v}\rangle| \le \|B_1(\mathbf{v},\mathbf{v})\|\|\Delta\mathbf{v}\|, \tag{B.10}$$

From Hölder inequality and Gagliardo-Nirenberg inequality we deuce that there $C_1 > 0$ such that

$$\|B_2(\mathbf{v},\mathbf{d})\| \le C_1\|\mathbf{v}\|^{1-\frac{n}{4}}\|\nabla\mathbf{v}\|^{\frac{n}{4}}\|\nabla\mathbf{d}\|^{1-\frac{n}{4}}\|\Delta\mathbf{d}\|^{\frac{n}{4}}, \text{ for any } \mathbf{v}\in\mathbb{V}, \mathbf{d}\in\mathbb{H}^2.$$

from which along with Eq. (B.10) we derive that

$$\begin{aligned}|\langle B_1(\mathbf{v},\mathbf{v}),\Delta\mathbf{v}\rangle| \leq& c\|\mathbf{v}\|^{1-\frac{n}{4}}\|\mathrm{A}_1^{\frac{1}{2}}\mathbf{v}\|^{\frac{n}{4}}\|\mathrm{A}_1^{\frac{1}{2}}\mathbf{v}\|^{1-\frac{n}{4}}\|\mathrm{A}_1\mathbf{v}\|^{\frac{n}{4}}\|\mathrm{A}_1\mathbf{v}\|,\\ \leq& c\|\mathbf{v}\|^{1-\frac{n}{4}}\|\mathrm{A}_1^{\frac{1}{2}}\mathbf{v}\|\|\mathrm{A}_1\mathbf{v}\|^{1+\frac{n}{4}}.\end{aligned} \tag{B.11}$$

Let us recall Young's inequality $ab \leq C(\alpha,p,q)a^p + \alpha b^q$ for $p^{-1}+q^{-1}=1$ and arbitrary $\alpha>0$. Let us choose $p=\frac{8}{n+4}$ and $q=\frac{8}{4-n}$. Applying Young's inequality with the above p and q in Eq. (B.11) we obtain

$$|\langle B_1(\mathbf{v},\mathbf{v}),\Delta\mathbf{v}\rangle| \leq \alpha\|\Delta\mathbf{v}\|^2 + C(\alpha,p,q)\|\mathbf{v}\|^2\|\mathrm{A}_1^{\frac{1}{2}}\mathbf{v}\|^{2+\frac{2n}{4-n}} \tag{B.12}$$

Let us look at the other terms. Thanks to the Assumption 2.1 it is not difficult to check that

$$G^2(\mathbf{d}) = (\mathbf{d}\times h)\times h \in D(A_2),$$

and

$$G(\mathbf{d}) = \mathbf{d}\times h \in D(A_2).$$

From the definition of Ψ' and these last remarks we easily deduce that

$$\begin{aligned}\frac{1}{2}\Psi'(\mathbf{d})[G^2(\mathbf{d})] =& \langle\Delta\mathbf{d}-f(\mathbf{d}),\Delta(\mathbf{d}\times h)\times h+(\mathbf{d}\times h)\times\Delta h - f'(\mathbf{d})(\mathbf{d}\times h)\times h\rangle,\\ =& \langle\Delta\mathbf{d}-f(\mathbf{d}),([\Delta\mathbf{d}-f(\mathbf{d})]\times h)\times h\rangle + \langle\Delta\mathbf{d}-f(\mathbf{d}),[f(\mathbf{d})+(\mathbf{d}\times\Delta h)]\times h\rangle\\ &+\langle\Delta\mathbf{d}-f(\mathbf{d}),(\mathbf{d}\times h)\times\Delta h - f'(\mathbf{d})(\mathbf{d}\times h)\times h\rangle,\\ =& -\|h\times(\Delta\mathbf{d}-f(\mathbf{d}))\|^2 + \langle\Delta\mathbf{d}-f(\mathbf{d}),[f(\mathbf{d})+(\mathbf{d}\times\Delta h)]\times h\rangle\\ &+\langle\Delta\mathbf{d}-f(\mathbf{d}),(\mathbf{d}\times h)\times\Delta h\rangle - \langle\Delta\mathbf{d}-f(\mathbf{d}),f'(\mathbf{d})(\mathbf{d}\times h)\times h\rangle,\\ =& -\|h\times(\Delta\mathbf{d}-f(\mathbf{d}))\|^2 + K_1+K_2+K_3.\end{aligned} \tag{B.13}$$

Owing to the assumption on h we have that for any $\delta_1>0$

$$\begin{aligned}|K_1| \leq& \delta_1\|\Delta\mathbf{d}-f(\mathbf{d})\|^2 + C(\delta_1)(\|f(\mathbf{d})\|^2+\|\mathbf{d}\times\Delta h\|^2\|h\|^2_{\mathbb{L}^\infty}),\\ \leq& \delta_1\|\Delta\mathbf{d}-f(\mathbf{d})\|^2 + C(\delta_1)(C+\|\mathbf{d}\|^2_{\mathbb{L}^4}\|\Delta h\|^2_{\mathbb{L}^4}\|h\|^2_{\mathbb{L}^\infty}),\\ \leq& \delta_1\|\Delta\mathbf{d}-f(\mathbf{d})\|^2 + C(\delta_1)(C+C(h)\|\nabla\mathbf{d}\|^2).\end{aligned} \tag{B.14}$$

We can use a similar argument to prove that for any $\delta_2>0$

$$|K_2| \leq \delta_2\|\Delta\mathbf{d}-f(\mathbf{d})\|^2 + C(\delta_2)(C+C(h)\|\nabla\mathbf{d}\|^2), \tag{B.15}$$

and for any $\delta_3>0$

$$|K_3| \leq \delta_3\|\Delta\mathbf{d}-f(\mathbf{d})\|^2 + C(\delta_3)(C+C(h)\|\mathbf{d}\|^2). \tag{B.16}$$

By combining the inequalities (B.13)-(B.16) we obtain that for any $\delta_i>0, i=1,2,3,$

$$\frac{1}{2}\Psi'(\mathbf{d})[G^2(\mathbf{d})] \leq -\|h\times(\Delta\mathbf{d}-f(\mathbf{d}))\|^2 + (\sum_{i=1}^{3}\delta_i)\|\Delta\mathbf{d}-f(\mathbf{d})\|^2 + C(\delta_1,\delta_2,\delta_3)(C+C(h)\|\nabla\mathbf{d}\|^2). \tag{B.17}$$

Now let us take a look at

$$\begin{aligned}\frac{1}{2}\Psi''(\mathbf{d})[G(\mathbf{d}),G(\mathbf{d})] =& \|\Delta G(\mathbf{d}) - f'(\mathbf{d})G(\mathbf{d})\|^2 - \langle\Delta\mathbf{d}-f(\mathbf{d}),f''(\mathbf{d})[G(\mathbf{d}),G(\mathbf{d})]\rangle,\\ :=& J_1+J_2.\end{aligned}$$

It is easy to show that for any $\delta_4>0$ we have

$$\begin{aligned}|J_2| \leq& \delta_4\|\Delta\mathbf{d}-f(\mathbf{d})\|^2 + C(\delta_4)\|G(d)\|_{\mathbb{L}^4}\\ \leq& \delta_4\|\Delta\mathbf{d}-f(\mathbf{d})\|^2 + C(\delta_4)(C+C(h)\|\nabla\mathbf{d}\|^2).\end{aligned} \tag{B.18}$$

To treat J_1 we use an argument which is similar to the one for $\frac{1}{2}\Psi'(\mathbf{d})[G^2(\mathbf{d})]$. More precisely, we have

$$\Delta G(\mathbf{d}) - f'(\mathbf{d})G(\mathbf{d}) = [\Delta\mathbf{d} - f(\mathbf{d})] \times h + f(\mathbf{d}) \times h + \mathbf{d} \times \Delta h - f'(\mathbf{d})(\mathbf{d} \times h).$$

Since $\|a+b\|^2 = \|a\|^2 + 2\langle a, b\rangle + \|b\|^2$ we infer that for any $\delta_5 > 0$ we have

$$J_1 \leq \|[\Delta\mathbf{d} - f(\mathbf{d})] \times h\|^2 + \delta_5\|h\|^2_{\mathbb{L}^\infty}\|\Delta\mathbf{d} - f(\mathbf{d})\|^2 + C(\delta_5)\|f(\mathbf{d}) \times h - f'(\mathbf{d})(\mathbf{d} \times h) + \mathbf{d} \times \Delta h\|^2.$$

By the same reasoning used to obtain estimate (B.14) we have the following inequality

$$J_1 \leq \|h \times (\Delta\mathbf{d} - f(\mathbf{d}))\|^2 + \delta_5\|h\|^2_{\mathbb{L}^\infty}\|\Delta\mathbf{d} - f(\mathbf{d})\|^2 + C(\delta_5)(C(h) + C(h)\|\nabla\mathbf{d}\|^2). \tag{B.19}$$

Owing to (B.18) and (B.19) we obtain

$$\begin{aligned}\frac{1}{2}\Psi''(\mathbf{d})[G(\mathbf{d}), G(\mathbf{d})] \leq \|h \times (\Delta\mathbf{d} - f(\mathbf{d}))\|^2 + (\delta_4 + \delta_5\|h\|^2_{\mathbb{L}^\infty})\|\Delta\mathbf{d} - f(\mathbf{d})\|^2 \\ + C(\delta_4, \delta_5)(C(h) + C(h)\|\nabla\mathbf{d}\|^2).\end{aligned} \tag{B.20}$$

With the help of BDG's inequality we have

$$2\mathbb{E}\sup_{0\leq s\leq t_k}\left|\int_0^s \Phi(r)\Psi'(\mathbf{d})[G(\mathbf{d})]dW_2(r)\right| \leq C\mathbb{E}\left(\int_0^{t_k}[\Phi(s)]^2|\Psi'(\mathbf{d})[G(\mathbf{d})]|^2 ds\right)^{\frac{1}{2}}.$$

Now invoking the definition of $\Psi'(\mathbf{d})[G(\mathbf{d})]$, Cauchy-Schwarz's inequality and Cauchy's inequality we have

$$\begin{aligned}2\mathbb{E}\sup_{0\leq s\leq t_k}\left|\int_0^s \Phi(r)\Psi'(\mathbf{d})[G(\mathbf{d})]dW_2(r)\right| &\leq \mathbb{E}\Bigg[\sup_{0\leq s\leq t_k}\sqrt{\Psi(s)}\|\Delta\mathbf{d}(s) - f(\mathbf{d}(s))\| \\ &\quad\times\left(\int_0^s \Phi(r)\|\Delta\Big((\mathbf{d} \times h) \times h\Big) - f'(\mathbf{d})\Big((\mathbf{d} \times h) \times h\Big)\|^2 dr\right)^{\frac{1}{2}}\Bigg] \\ &\leq C(\delta_6)\mathbb{E}\int_0^{t_k}\Phi(s)\|\Delta\Big((\mathbf{d} \times h) \times h\Big) - f'(\mathbf{d})\Big((\mathbf{d} \times h) \times h\Big)\|^2 ds \\ &\quad + \delta_6\mathbb{E}\sup_{0\leq s\leq t}\Phi(s)\|\Delta\mathbf{d} - f(\mathbf{d})\|^2,\end{aligned}$$

for any $\delta_6 > 0$. Thanks to this last inequality and Eqs. (B.14)-(B.16) it is easy to prove that

$$\begin{aligned}2\mathbb{E}\sup_{0\leq s\leq t}\left|\int_0^{t_k}\Phi(s)\Psi'(\mathbf{d})[G(\mathbf{d})]dW_2(s)\right| \leq C(\delta_6, h)\mathbb{E}\int_0^{t_k}\Psi(s)(C + C(h)\|\nabla\mathbf{d}\|^2)ds \\ + C(\delta_6, h)\mathbb{E}\int_0^{t_k}\Phi(s)\|\Delta\mathbf{d} - f(\mathbf{d})\|^2 ds + \delta_6\mathbb{E}\sup_{0\leq s\leq t_k}\Phi(s)\|\Delta\mathbf{d} - f(\mathbf{d})\|^2.\end{aligned} \tag{B.21}$$

With similar idea we can prove that

$$2\mathbb{E}\sup_{0\leq s\leq t_k}\left|\int_0^s \Phi(r)\langle \mathrm{A}_1^{\frac{1}{2}}S(\mathbf{v}), \mathrm{A}_1^{\frac{1}{2}}\mathbf{v}\rangle dW_1(r)\right| \leq \alpha_1\mathbb{E}\sup_{0\leq s\leq t_k}\Phi(s)\|\mathrm{A}_1^{\frac{1}{2}}\mathbf{v}\|^2 + \mathbb{E}\int_0^{t_k}\Phi(s)\|S(\mathbf{v})\|^2_{\mathcal{J}_2(\mathrm{K}_1,\mathrm{V})}ds. \tag{B.22}$$

Let $\alpha = 1$ and $\alpha_1 = \delta_6 = \frac{1}{2}$. For the function $\Psi(s)$ in Eq. (B.8) let us take

$$\phi(s) = C(1, p, q)\|\mathbf{v}\|^2\|\mathrm{A}_1^{\frac{1}{2}}\mathbf{v}\|^{\frac{2n}{4-n}},$$

where the constant $C(1, p, q)$ is defined in (B.12). The application of Itô's formula to the real-valued stochastic processes $\Phi(t)\Big(\|\mathrm{A}_1^{\frac{1}{2}}\mathbf{v}\|^2 + \Psi(\mathbf{d})\Big)$ and the estimates (B.12), Eq. (B.17), Eq.

(B.20), Eq. (B.21) and Eq. (B.22) yield that

$$\mathbb{E}\Phi(t_k)\Big(\|\mathrm{A}_1^{\frac{1}{2}}\mathbf{v}(t_k)\|^2+\Psi(\mathbf{d}(t_k))\Big)+\mathbb{E}\int_0^{t_k}\Phi(s)\Big(2\|\mathrm{A}_1\mathbf{v}\|^2+\|\nabla(\Delta\mathbf{d}-f(\mathbf{d}))\|^2\Big)ds$$
$$\leq\|\nabla\mathbf{v}_0\|^2+\Psi(\mathbf{d}_0)+\mathbb{E}\int_0^{t_k}\Phi(s)\|S(\mathbf{v})\|^2_{\mathcal{J}_2(\mathrm{K}_1,\mathrm{V})}ds+C(h)\mathbb{E}\int_0^{t_k}\Phi(s)\|\Delta\mathbf{d}-f(\mathbf{d})\|^2ds$$
$$+C\mathbb{E}\int_0^{t_k}\Phi(s)(C+C(h)\|\nabla\mathbf{d}\|^2)ds.$$

The hypotheses on S in Assumption 2.1 yields that

$$\mathbb{E}\Big[e^{-\int_0^{t_k}\phi(r)dr}\Big(\|\mathrm{A}_1^{\frac{1}{2}}\mathbf{v}(t_k)\|^2+\Psi(\mathbf{d}(t_k))\Big)\Big]+\mathbb{E}\int_0^{t_k}\Phi(s)\Big(2\|\mathrm{A}_1\mathbf{v}\|^2+\|\nabla(\Delta\mathbf{d}-f(\mathbf{d}))\|^2\Big)ds$$
$$\leq\|\nabla\mathbf{v}_0\|^2+\Psi(\mathbf{d}_0)+C(h)\mathbb{E}\int_0^{t_k}\Phi(s)\Big(\|\nabla\mathbf{v}\|^2+\|\Delta\mathbf{d}-f(\mathbf{d})\|^2\Big)ds$$
$$+C(\mathbf{v}_0,\mathbf{d}_0)e^{C(h,2)t_k}t_k. \tag{B.23}$$

Here we have used Proposition B.1 to infer that there exists a constant $C(\mathbf{v}_0,\mathbf{d}_0)>0$ such that

$$\sup_{0\leq s\leq t_k}\mathbb{E}\Big(\|\mathbf{v}(s)\|^2+\|\nabla\mathbf{d}(s)\|^2\Big)\leq C(\mathbf{v}_0,\mathbf{d}_0)e^{C(h,2)t_k}t_k,$$

for any $t>0$. From (B.23) and Gronwall's inequality we derive Eq. (B.3). □

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DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF YORK, HESLINGTON, YORK YO10 5DD, UK
E-mail address: zdzislaw.brzezniak@york.ac.uk

DEPARTMENT OF MATHEMATICS AND INFORMATION TECHNOLOGY, MONTANUNIVERSITÄT LEOBEN, FR. JOSEF-STR. 18, 8700 LEOBEN, AUSTRIA
E-mail address: erika.hausenblas@unileoben.ac.at
E-mail address: paul.razafimandimby@unileoben.ac.at